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Conditioning a Segmented Stem Profile Model for Two Diameter Measurements

Raymond L. Czaplewski and Joe P. McClure

ABSTRACT. The stem profile model of Max and Burkhart (1976) is conditioned for dbh and a second upper stem measurement. This model was applied to a loblolly pine data set using diameter outside bark at 5.3m (i.e., height of 17.3 foot Girard form class) as the second upper stem measurement, and then compared to the original, unconditioned model. Variance of residuals was reduced; however, bias was approximately the same for both the conditioned and unconditioned models. Benchmark prediction problems, which reflect common multiproduct utilization criteria for southern pines, were used to judge the practical importance of these reductions. Square root of the average squared residuals from the conditioned model were 10 to 25% less than those from the unconditioned model for most benchmark evaluations. These reductions might be of practical importance, depending on the particular objectives and the relative cost of an upper stem measurement compared to the cost of measuring more trees. FOR. SCI. 34(2): 512–522.

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CAO, ET AL. (1980) evaluated six stem profile models for loblolly pine (*Pinus taeda*). The segmented model of Max and Burkhart (1976), which has six parameters and complex geometry, was best for predicting diameters inside bark. The Ormerod (1973) model was the second best; it has three parameters and is geometrically much simpler. One difference between these two models is predicted diameter at breast height (dbh). The Ormerod model for diameter outside bark is mathematically formulated (conditioned) to exactly predict observed dbh. Even though the Max and Burkhart model contains dbh, the predicted dbh does not necessarily equal the measured dbh outside bark. This is not necessarily disconcerting for outside bark models because dbh measurements can have a positive bias if a diameter tape is used (Avery and Burkhart 1983). However, Cao, et al. (1980) further conditioned the Omerod model for inside bark predictions by introducing an explicit submodel for bark thickness at breast height. Based on our interpretation of these results, conditioning an inside bark stem profile model for dbh using a bark submodel might improve model predictions. Measurement of a second diameter farther up the stem might further improve estimates, at least near this second measurement (Bruce, et al. 1968, Matney and Sullivan 1979). Therefore, different versions of the Max and Burkhart stem profile model where conditioned for dbh and an upper stem measurement using explicit bark submodels, and performance was compared to the original, unconditioned formulation.

DATA

Data used in this study have been gathered by the USDA Forest Service, Southeastern Forest Experiment Station, Forest Inventory and Analysis Project (SE FIA), at Asheville, NC, over the past 20 years from Florida, Georgia, North Carolina, South Carolina, and Virginia. Methods are fully described by Cost (1978) and USDA Forest Service (1986). Diameters were measured at ground, stump (0.3m), breast

The authors are, respectively, Mathematical Statistician, Multiresource Inventory Techniques, USDA Forest Service, Rocky Mountain Forest and Range Experiment Station, Fort Collins, CO 80526; and Project Leader, Forest Inventory and Analysis, USDA Forest Service, Southeastern Forest Experiment Station, Asheville, NC 28804. Original Forest Science manuscript number 3909. Manuscript received May 5, 1986.

height (1.37m), height of Girard form class (5.3m), 1.2m intervals above the stump to a 17.5cm diameter outside bark (dob), and 1.5m intervals to a 10cm dob. Forty-two percent of the trees were felled during commercial logging; diameters were measured with calipers, and heights were measured with steel tapes. Bark thickness was measured twice with a Swedish bark gauge, and was used to estimate diameter inside bark (dib). The remaining data are from standing trees; diameters were measured using a steel tape or optical dendrometer, and heights measured with sectional aluminum poles. Only loblolly pine trees within a 17.5 to 50cm dbh range were used. Smaller trees have limited potential for multiproduct utilization. Larger trees are rare and often unavailable for commercial harvest. A few trees with major defects, broken tops, or excessive limbing also were excluded because of their unmerchantable or atypical characteristics. To more rigorously isolate the effect of alternative model formulations, rather than variability in stem taper, only those trees with a 40 to 49% crown ratio (Cost 1978) were used because of the evidence for a relationship between crown ratio and changes in stem form within the live crown (Larson 1963, Dell 1979, Kilkki and Varmola 1981). Measurements at ground level were not used because they were suspected of degrading model performance in the lower mainstem (McClure and Czaplewski 1986), and dimensions below stump level are less important than the lower bole for commercial utilization standards. The distribution of dbh and tree heights are in Table 1. All measurement plots were randomly divided into two data sets: (1) a single, developmental data set used to estimate parameters in all formulations of the taper models; and (2) an independent test sample used to compare performance among models. The former contains 1,887 tree sections from 144 trees; the latter contains 3,641 tree sections from 276 trees. The test data contains more trees so that variance of prediction errors can be better estimated.

METHODS

The Max and Burkhart (1976) regression model for stem profile is:

$$(\hat{d}/D)^2 = b_1(h/H-1) + b_2(h^2/H^2-1) + b_3(a_1-h/H)^2I_1 + b_4(a_2-h/H)^2I_2 \quad (1)$$

where

$\hat{d}$ = estimated diameter at height h;

D = diameter at breast height (dob);

b_i = regression coefficients estimated from sample data $i = \{1,2,3,4\}$;

h = height above ground to estimated top diameter $\hat{d}$;

H = total tree height;

a_i = join point parameters estimated from sample data $i = \{1,2\}$;

$$I_i = \begin{cases} 1, \text{ for } h/H < a_i; \; i = \{1,2\} \\ 0, \text{ for } h/H \geq a_i. \end{cases}$$

Equations for estimating diameter, height, and cubic volume using this model were derived by Cao (1978).

Conditioning the Model

Three of the six parameters in (1) were considered for conditioning the model to exactly predict observed dbh: b_1, b_2, b_3. (The value of the conditioned parameter is recomputed for each tree so that predicted and observed dbh are identical.) The join points (a_1,a_2) were not conditioned because they would introduce difficult nonlinear problems, and b_4 was not used because breast height may be above the lower join point, where b_4 has no effect on predicted diameters. Therefore, three models conditioned for dbh were studied.

TABLE 1. Number of trees in the data set by dbh and height classes. Merchantable height is height to a 10cm top dob.

	Number of trees by total height (m) classes				Average height (m)	
dbh (cm)	10 to 15	15 to 20	20+	Total	Total	Merchantable
17.5 to 22.5	76	37	0	113	14.3	9.6
22.5 to 27.5	19	58	18	95	17.1	12.9
27.5 to 32.5	4	43	25	72	19.1	15.3
32.5 to 37.5	2	32	41	75	20.3	16.4
37.5 to 42.5	0	6	30	36	22.1	18.4
42.5 to 50.0	0	5	24	29	22.0	18.6
Total	101	181	138	420	18.0	13.9

When an additional upper-stem measurement is available, it is possible to further condition the stem profile model to exactly predict this second measurement. The same three parameters (b_1, b_2, b_3) were considered for simultaneous conditioning to two stem measurements. However, b_3 was not conditioned for upper-stem measurements because it might not affect model predictions in such areas of the bole. Therefore, an additional four models were investigated: (1) b_1 conditioned for dbh and b_2 conditioned for the upper-stem measurement; (2) b_1 for the upper-stem and b_2 for dbh; (3) b_1 for the upper-stem and b_3 for dbh; and (4) b_2 for the upper-stem and b_3 for dbh.

INSIDE BARK MODELS

Inside bark predictions are usually made with outside bark measurements at breast height (and perhaps a measured dob from the upper stem). An estimate of bark thickness is required to condition an inside bark model for outside bark measurements. This can be done with a bark gauge and used directly with the models in the Appendix. However, bark gauges are not used in many inventories, and attention was focused on models that predict dib using observed dob.

The following models were used to estimate dib at breast height ($\hat{d}_D$) and dib at form class height ($\hat{d}_u$) using measurements of dbh (D) and dob at an upper bole position (D_u):

$$(\hat{d}_D/D)^2 = b_5 D^{-2} + b_6 D^{-1} + b_7 ; \quad (2)$$

$$(\hat{d}_u/D)^2 = b_8 D^{-2} + b_9 (D_u D^{-2}) + b_{10} (D_u^2 D^{-2}) . \quad (3)$$

These are simple models for dib, where dib is a linear function of dob (see Appendix). In addition, more complex linear models for $(\hat{d}_D/D)^2$ and $(\hat{d}_u/D)^2$ were also evaluated that included the following independent variables: H, $1/H$, D/H, D^2/H^2, D_u^2/D^2, $D_u/(D^2H)$.

Parameter estimates for these bark models could have been made, and then parameters for the stem profile model estimated in a second step. However, the Max and Burkhart model was conditioned so that parameters in the stem profile model and the bark thickness submodels could be estimated simultaneously (see Appendix). This was expected to minimize residual error of the inside bark stem profile model.

ESTIMATING PARAMETERS

The regression models for stem profile are linear relative to parameters b_i; however, they are nonlinear relative to the join points (a_1, a_2). Preliminary estimates of the join points were made by incrementally exploring a wide range of reasonable values ($0 < a_2 < a_1 < 1$), and using multiple linear regression to estimate the remaining parameters. The residual mean square error ($s^2_{y \cdot x}$) from these regressions was used to select the best initial estimate of a_1 and a_2. This initial step assured that a global minimum was achieved. (No local minima were encountered.) These estimates of join points were further refined using the ZXMIN algorithm, which is a nonlinear minimization routine (IMSL 1972). The join points were simultaneously varied by ZXMIN to minimize the $s^2_{y \cdot x}$ from the multiple linear regression.

Evaluating Models

The three models conditioned for dbh, and the four models conditioned for both dbh and dob at 5.3m, were initially compared using the mean squared residual for relative diameter estimates (observed minus predicted *d*/*D*) in Table 2. Also, medians and quartiles of the residual error of relative diameter estimates were plotted as a function of relative height. The plots were visually evaluated using minimum deviation of the median from zero as criterion to select the best models conditioned for dbh, and dbh plus dob at 5.3m. Emphasis was placed on performance in the lower-stem because it is more commercially important. Both evaluations used independent test data.

Inside-bark models were further evaluated using performance in 12 benchmark prediction problems. The benchmarks relate to common merchantability standards for peeler and saw logs from southern pines, and were used to judge the practical importance of differences between models. These benchmarks are: diameter at 5.3, 10.3, and 15.3m above ground; height to 20, 15, and 10cm dib; 1.2m section volumes above the 0.3, 5.3, and 10.3m heights; volume between stump height (0.3m) and a 20-cm dib; and volume between 20 to 15, and 15 to 10cm dib.

The mean and standard deviation of independent prediction errors for these benchmarks were calculated for each 2.5cm dbh class, and were used to compare performance of models as a function of tree size. Methods described by Reynolds (1984) were used to test hypotheses ($\alpha = 0.05$) that residuals in the benchmark predictions were normally distributed within each dbh class. For those dbh classes in which the normality hypothesis was not rejected, the hypothesis of unbiased residuals (mean error = 0) was also tested ($\alpha = 0.05$). All benchmark tests of hypothesis were performed using the independent test sample that included 20 to 40 trees in each dbh class.

RESULTS AND DISCUSSION

Compared to the original Max and Burkhart model, the standard deviation of residuals for the models conditioned for dbh, but not for dob at 5.3m, were 2 to 8% less for predictions of relative dib, and 3 to 5% less for predictions of relative dob (Table 2). Choice of conditioned parameter (b_1, b_2, or b_3) made little difference. However, the model that was conditioned using b_3 had slightly less mean residual error in the lower stem. The simple bark models produced more precise estimates of relative dib over the entire main stem than the more complex models (Table 2); this was true for the independent test data, but not for the data used in parameter estimation.

For inside bark models, the standard deviation of residual error in relative diameter estimates, conditioned for both dbh and dob at 5.3m, were 5 to 11% less than those for the unconditioned model (Table 2); however, they were 0.94 to 2.3 times as large for estimates of relative dob. Most of the differences in residual error occurred in the upper half of the main stem, which is less valuable for multiproduct utilization than the lower sections. Mean residual error in the lower stem was smallest for the model in which b_1 and b_2 were conditioned for dbh and dob at 5.3m. The complex bark submodels had standard deviations of residual errors (Table 2) that were 0.94 to 1.03 times as large as those of the simple bark submodels, as evaluated using independent test data. Most of the differences in performance between bark submodels occurred in the upper stem.

Benchmark Diameter Estimates

Twelve benchmark prediction problems were used to judge the practical importance of differences between the original model and a conditioned model (conditioned for dbh and dob at 5.3m using b_1 and b_2 and the simple bark submodels). The first set of

TABLE 2. Comparison of square root of the mean squared residuals (units = d/D), using independent test data, for alternate methods to condition the Max and Burkhart model.

	Outside bark	Inside bark: Simple bark submodel	Inside bark: Complex bark submodel
Unconditioned	0.0778	0.0747[a]	0.0747[a]
Conditioned for dbh			
b_1 conditioned for dbh	0.0754	0.0693	0.0735
b_2 conditioned for dbh	0.0736	0.0690	0.0714
b_3 conditioned for dbh	0.0742	0.0691	0.0714
Conditioned for dbh and form class			
b_1 conditioned for dbh, b_2 for form class	0.1798	0.0709	0.0666
b_2 conditioned for dbh, b_1 for form class	0.1798	0.0709	0.0666
b_3 conditioned for dbh, b_1 for form class	0.0732	0.0667	0.0690
b_3 conditioned for dbh, b_2 for form class	0.0776	0.0679	0.0688

[a] These stem profile models have no explicit bark submodel.

benchmarks used dib at the top of the first three 5m logs. There was little difference between models in statistically significant bias by dbh class for all three diameter predictions; however, there were large differences in the standard deviation of residuals (Table 3). The conditioned model was consistently more precise in predicting dib at the top of the first log. Standard deviation of residual errors ranged from one-half (for 17.5cm dbh trees) to one-fourth (for 50cm dbh trees). There were fewer differences in predicting dib at the top of the second 5m log. Precision of both models was approximately equal for trees with a dbh of 30cm or less; however, standard deviation of residual error of the conditioned model was one-fourth less for larger trees. There were no major differences between models in predicting dib at the top of the third 5m log.

Benchmark Height Estimates

The second set of benchmarks tested predictions of height to three different top diameters. There were few differences between the two models in statistically significant bias; however, there were differences in the standard deviation of prediction error (Table 3). Standard deviation of errors in predicting height to a 20cm dib was one-fourth less for the conditioned model compared to the unconditioned model. There were fewer differences between models in predicting height to a 15cm dib. Standard deviation of residuals in predicting this height were 10% less for the conditioned model for trees with a dbh of 35cm or less; errors for larger trees were very similar for both models. For trees with a dbh of 35cm or less, there were no major differences between models in predicting height to a 10cm top dib. For larger trees, the standard deviation of residuals for this height using the unconditioned model was 25 to 50% less than the conditioned model. This was the only benchmark for which the unconditioned model was superior to the conditioned model.

Benchmark Volume Estimates

The third set of benchmark prediction problems used cubic volume in 1.2m sections at the base of the first three 5m logs. Again, there was virtually no difference between models in statistically significant mean residual error. Also, there was little difference between the two models in predicting cubic volume between 0.3m and 1.5m (Table 3). However, the conditioned model was consistently more precise than

TABLE 3. Approximate decrease in standard deviation of benchmark residual errors from a stem profile model conditioned for dbh and dob at 5.3m compared to unconditioned model.

Benchmark	dbh	Decrease in standard deviation of residuals (%)
Estimated dob at		
5.3m height	17.5cm	50
	50cm	25
10.3m height	under 30cm	0
	over 30cm	25
15.3m height	all trees	0
Estimated height to		
20cm dib	all trees	25
15cm dib	under 35cm	10
	over 35cm	0
10cm dib[a]	under 35cm	0
	35cm	−25
	50cm	−50
Estimated cubic volume		
stump to 1.5m height	all trees	0
5.3 to 6.5m height	17.5cm	50
	50cm	25
10.3 to 11.5m height	under 38cm	10
	over 38cm	25
10 to 15cm dib	all trees	0
15 to 20cm dib	all trees	0
stump to 20cm dib	17.5cm	10
	50cm	40

[a] This is only benchmark for which the conditioned model was less precise than unconditioned model.

the unconditioned model when used to predict volume of the other two 1.2m sections. This was the case for the entire dbh range. The conditioned model had standard deviations of residuals for volume between 5.3 and 6.5m which ranged from 50% less for the smallest trees (17.5cm dbh) to 25% less for the largest trees (50cm dbh). For predicting volume between 10.3 and 11.5m, the standard deviation of residuals from the conditioned model were 10 to 25% less than those of the unconditioned model; the greatest improvements were realized for trees with a dbh of 38 to 50cm.

The final set of benchmark prediction problems tested performance of the two stem profile models in predicting cubic volume between specified dib's. There were few differences between models in statistically significant mean residual error. Also, both models were equally precise, for all tree sizes, in predicting volume between 10 and 15cm dib, and between 15 and 20cm dib (Table 3). However, the conditioned model was superior to the unconditioned model in predicting volume between stump height and 20cm dib. Standard deviation of errors from the conditioned model were 10 to 40% less than those from the unconditioned model, increasing proportional to tree size. Residuals for volume estimates between stump height and a top diameter are typically large (e.g., standard deviations of 0.1 to 0.2m^3 for volume below a 20cm dib) because it requires estimating height to that top diameter, and a small error in estimating height can produce a large error in estimating volume below that height.

LITERATURE CITED

AVERY, T. E., and H. E. BURKHART. 1983. Forest measurements. McGraw-Hill, New York. 331 p.

BRUCE, D., R. O. CURTIS, and C. VANCOEVERING. 1968. Development of a system of taper and volume tables for red alder. For. Sci. 14:339–350.

CAO, V. Q. 1978. Prediction of cubic-foot volume of loblolly pine to any top diameter limit and to any point on tree bole. MS thesis, Va. Polytech. Inst. and State Univ., Blacksburg, VA. 67 p.

CAO, V. Q., H. E. BURKHART, and T. A. MAX. 1980. Evaluation of two methods for cubic-volume prediction of loblolly pine to any merchantable limit. For. Sci. 26:71–80.

COST, N. D. 1978. Multiresource inventories—a technique for measuring volumes in standing trees. USDA For. Serv. Res. Pap. SE-196. 18 p.

DEMAERSCHALK, J. P., and A. KOZAK. 1977. The whole-bole system: A conditioned dual-equation system for precise prediction of tree profies. Can. J. For. Res. 7:488–497.

DELL, T. R. 1979. Potential of using crown ratio in predicting product yield. P. 843–851 *in* Forest resource inventories, W. E. Frayer, ed. Colorado State Univ., Fort Collins.

IMSL. 1972. International mathematical and statistical library: GNB Building, 7500 Bellaire Blvd., Houston, TX.

KILKKI, P., and M. VARMOLA. 1981. Taper curve models for Scots Pine and their applications. Acta For. Fenn. 174:1–60.

LARSON, P. R. 1963. Stem form development of forest trees. For. Sci. Monogr. 5:1–42.

MATNEY, T. G., and A. D. SULLIVAN. 1979. Absolute form quotient taper curves and their application to old-field plantation loblolly pine trees. P. 831–842 *in* Forest resource inventories, W. E. Frayer, ed. Colorado State Univ., Fort Collins.

MAX, T. A., and H. E. BURKHART. 1976. Segmented polynomial regression applied to taper equations. For. Sci. 22:283–289.

MCCLURE, J. P., and R. L. CZAPLEWSKI. 1986. Compatible taper equation for loblolly pine. Can. J. For. Res. 16:1272–1277.

ORMEROD, D. W. 1973. A simple bole model. For. Chron. 49:136–138.

REYNOLDS, M. 1984. Estimating the error in model predictions. For. Sci. 30:454–469.

USDA Forest Service. 1986. Field instructions for the Southeast. Forest Inventory and Analysis Work Unit, SE For. Exp. Stn., Asheville, NC.

APPENDIX

CONDITIONING b_3 FOR dbh

To condition b_3 so that predicted dbh ($\hat{d}_D$) equals measured dbh (D) at breast height (H_D), let

$$\hat{d}_D/D = 1$$

$$I_{2,D} = \begin{cases} 0, & \text{for } H_D/H \geq a_2 \\ 1, & \text{for } H_D/H < a_2. \end{cases}$$

It is assumed that $H_D/H < a_1$ (i.e., breast height is below the top join-point), which makes $I_{1,D} = 1$. Solving the unconditioned model (1) for b_3 produces

$$b_3 = [(\hat{d}_D/D)^2 + b_1(1 - H_D/H) + b_2(1 - H_D^2/H^2) - b_4(a_2 - H_D/H)^2 I_{2,D}](a_1 - H_D/H)^{-2} \quad (4)$$

Substituting b_3 in equation (4) into the original model (1) yields

$$(\hat{d}/D)^2 = b_1X_1 + b_2X_2 + b_4X_3 + W \quad (5)$$

where

$$X_1 = h/H - 1 + (1 - H_D/H)(a_1 - H_D/H)^{-2}(a_1 - h/H)^2 I_1;$$
$$X_2 = h^2/H^2 - 1 + (1 - H_D^2/H^2)(a_1 - H_D/H)^{-2}(a_1 - h/H)^2 I_1;$$
$$X_3 = (a_1 - h/H)^2 I_2 - (a_2 - H_D/H)^2(a_1 - H_D/H)^{-2}(a_1 - h/H)^2 I_1 I_{2,D};$$

$W = (a_1 - h/H)^2(a_1 - H_D/H)^{-2}I_1$;

b_i = parameters in the original model (1), $i = \{1,2,4\}$.

The above formulation is designed for outside bark models or inside bark models when bark thickness is directly observed (e.g., $\hat{d}_D/D$ is measured directly using a diameter tape and a bark gauge). However, dib is usually predicted using observed dob. In this case, $(\hat{d}_D/D)^2$ must be estimated, and substituting equation (4) into the original model (1) produces

$$(\hat{d}/D)^2 = b_1X_1 + b_2X_2 + b_4X_3 + W(\hat{d}_D/D)^2. \tag{6}$$

A common estimator for dib at breast height given dbh outside bark is the simple linear model:

$$\hat{d}_D = c_0 + c_1D. \tag{7}$$

This relationship was used by Cao (1978) for the stem profile model of Ormerod (1973). It is also used with (6) to complete the formulation of the segmented conditioned model for predicting inside bark dimensions.

A two-step process could have been employed in which parameters for predicting $(\hat{d}_D/D)^2$ are first estimated using a quadratic transformation of model (7):

$$(\hat{d}_D/D)^2 = b_5D^{-2} + b_6D^{-1} + b_7. \tag{8}$$

In the second step, b_i, $i = \{1,2,4\}$ could be estimated using (6) and (8) in the following regression model:

$$(\hat{d}/D)^2 - W(\hat{d}_D/D)^2 = b_1X_1 + b_2X_2 + b_4X_3. \tag{9}$$

However, it is possible to simultaneously estimate all parameters in (8) and (9) using the following regression model with a zero intercept:

$$(\hat{d}/D)^2 = b_1X_1 + b_2X_2 + b_4X_3 + b_5X_4 + b_6X_5 + b_7W \tag{10}$$

where

X_i = transformed independent variables in conditioned taper model (5), $i = \{1,2,3\}$;

$X_4 = W/D^2$;

$X_5 = W/D$;

W = a function of h/H, defined in the conditioned model (5);

b_i = estimated regression coefficients in the original model (1) for $i = \{1,2,4\}$, and in the inside-bark model (8) for $i = \{5,6,7\}$

Simultaneously estimating all parameters was expected to minimize the residual error. Also, variance for $(\hat{d}/D)^2$ as a function of relative height (h/H) in (10) was found to be nearly homogeneous by Bruce, et al. (1968) for red alder, by Demaerschalk and Kozak (1977) for maple and Douglas fir, and by McClure and Czaplewski (1986) for loblolly pine.

Using the parameters estimated by (10) to subsequently predict diameters can be simplified as follows. If the taper model predicts dob, or bark thickness at breast height is directly measured, then $(\hat{d}_D/D)^2$ is a known constant. Otherwise, $(\hat{d}_D/D)^2$ is estimated for each tree using bark model (8). The value of the conditioned parameter (b_3) is then computed for each tree using (4); any diameter, height, and volume predictions for this one tree may then be made using the original model (1) and formulae given by Cao (1978).

Additional Upper-Stem Measurement

To simultaneously condition b_1 for dbh and b_2 for the upper-stem measurement, let

$$I_{1,D} = \begin{cases} 0, \text{ for } H_D/H \geq a_1 \\ 1, \text{ for } H_D/H < a_1 \end{cases}$$

$$I_{2,D} = \begin{cases} 0, \text{ for } H_D/H \geq a_2 \\ 1, \text{ for } H_D/H < a_2 \end{cases}$$

$$I_{1,u} = \begin{cases} 0, \text{ for } H_u/H \geq a_1 \\ 1, \text{ for } H_u/H < a_1 \end{cases}$$

$$I_{2,u} = \begin{cases} 0, \text{ for } H_u/H \geq a_2 \\ 1, \text{ for } H_u/H < a_2 \end{cases}$$

where H_u is the height at which the upper-stem measurement is taken. Simultaneously solving the unconditioned model (1) for b_1 and b_2 results in

$$b_2 = [b_1(H_u/H - 1) + b_3(a_1 - H_u/H)^2 I_{1,u} + b_4(a_2 - H_u/H)^2 I_{2,u} - (\hat{d}_u/D)^2]/(1 - H_u^2/H^2) \quad (11)$$

$$b_1 = \left\{ \begin{array}{l} (\hat{d}_D/D)^2(H_u^2/H^2 - 1) + (\hat{d}_u/D)^2(1 - H_D^2/H^2) \\ + b_3[(H_D^2/H^2 - 1)(a_1 - H_u/H)^2 I_{1,u} + (1 - H_u^2/H^2)(a_1 - H_D/H)^2 I_{1,D}] \\ + b_4[(H_D^2/H^2 - 1)(a_2 - H_u/H)^2 I_{2,u} + (1 - H_u^2/H^2)(a_2 - H_D/H)^2 I_{2,D}] \end{array} \right\} / V \quad (12)$$

where

$$V = [(1 - H_D/H)(1 - H_u^2/H^2) - (1 - H_D^2/H^2)(1 - H_u/H)].$$

Substituting (11) and (12) into the original model (1) yields

$$(\hat{d}/D)^2 = b_3 X_6 + b_4 X_7 + W_D(\hat{d}_D/D)^2 + W_u(\hat{d}_u/D)^2, \quad (13)$$

where

$$\begin{aligned} X_6 = {} & (a_1 - h/H)^2 I_1 + (h/H - 1)\,[(1 - H_u^2/H^2)(a_1 - H_D/H)^2 I_{1,D} \\ & - (1 - H_D^2/H^2)(a_1 - H_u/H)^2 I_{1,u}]/V + (h^2/H^2 - 1)\,[(1 - H_D/H)(a_1 - H_u/H)^2 I_{1,u} \\ & - (1 - H_u/H)(a_1 - H_D/H)^2 I_{1,D}]/V; \end{aligned}$$

$$\begin{aligned} X_7 = {} & (a_2 - h/H)^2 I_2 + (h/H - 1)\,[(1 - H_u^2/H^2)(a_2 - H_D/H)^2 I_{2,D} \\ & - (1 - H_D^2/H^2)(a_2 - H_u/H)^2 I_{2,u}]/V + (h^2/H^2 - 1)\,[(1 - H_D/H)(a_2 - H_u/\mathrm{H})^2 I_{2,u} \\ & - (1 - H_u/H)(a_2 - H_d/H)^2\, I_{2,D}]/V; \end{aligned}$$

$$W_D = [(1 - h/H)(1 - H_u^2/H^2) - (1 - h^2/H^2)(1 - H_u/H)]/V;$$

$$W_u = [(1 - h^2/H^2)(1 - H_D/H) - (1 - h/H)(1 - H_D^2/H^2)]/V;$$

b_i = parameters in the original model (1), $i = \{3,4\}$;

V = constant defined in equation (12).

When d_D and d_u are measured directly, $\hat{d}_D/D$ and $\hat{d}_u/D$ are known constants (assuming negligible measurement error). However, dib is often predicted using dob measurements, and d_D and d_u must be estimated when inside bark dimensions are being predicted. Bark model (7) was used for $\hat{d}_D$, while a similar model was employed for $\hat{d}_u$. This produced the following regression model for inside bark dimensions:

$$(\hat{d}_u/D)^2 = b_8(D^{-2}) + b_9(D_u D^{-2}) + b_{10}(D_u^2 D^{-2}) \quad (14)$$

where $\hat{d}_u$ is estimated dib, and D_u is measured dob, at the upper stem position. Regression model (8) was selected for $(\hat{d}_D/D)^2$. However, simultaneous estimation of parameters in both the inside bark stem profile models and the bark models was

TABLE 4. Parameter estimates for taper models and bark submodels.

							Bark submodel					
	Segmented taper model						Breast height			Form class height		
Model type	a_1	a_2	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
Inside bark												
Unconditioned	0.6437	0.0984	−2.517	1.098	−1.660	66.01						
Conditioned for dbh	0.6586	0.0758	−2.478	1.072	*	118.58	−10.33	1.509	0.6811			
Conditioned for dbh and form class	0.6596	0.0878	*	*	−1.791	85.39	−33.65	5.546	0.5263	7.997	−37.81	0.3029
Conditioned for dbh inside bark	0.6598	0.0908	−3.453	1.517	*	103.26						
Outside bark												
Unconditioned	0.7100	0.1040	−3.210	1.418	−1.570	72.32						
Conditioned for dbh	0.6966	0.0933	−2.659	1.093	*	86.32						
Conditioned for dbh and form class	0.7667	0.0950	*	*	−1.997	82.95						

* These parameters calculated once for each tree to condition taper model to observed measurements.

performed using the following multiple linear regression model with zero intercept rather than (8), (13), and (14):

$$(\hat{d}/D)^2 = b_3X_6 + b_4X_7 + b_5X_4 + b_6X_5 + b_7W_D + b_8X_8 + b_9X_9 + b_{10}X_{10} \quad (15)$$

where,

X_i = transformed independent variables in conditioned model (13); $i = \{6,7\}$; and breast height bark model (8), $i = \{4,5\}$;

$X_8 = W_u/D^2$;

$X_9 = W_uD_u/D^2$;

$X_{10} = W_uD_u^2/D^2$;

W_D, W_u = functions of h/H, defined in (13);

b_i = parameters in original profile model (1), $i = \{3,4\}$; breast height bark model (8) parameters, $i = \{5,6,7\}$; and upper-stem bark model (14) parameters, $i = \{8,9,10\}$.

When applying the inside bark models with parameters estimated using (15), $(\hat{d}_D/D)^2$ is calculated for each tree using (8), and $(\hat{d}_u/D)^2$ using (14). These estimates are used to compute the values of the conditioned parameters (b_1 and b_2) for each tree using (12) and (11). These conditioned parameters are used with estimated values of b_3 and b_4 from (15) in the original model (1) to predict diameters, heights, and volumes using analytically derived geometric formulae (Cao 1978).

Estimated parameters for the original Max and Burkhart taper model, the model conditioned for dbh, and the model conditioned for both dbh and Girard form class (measured ob) are given in Table 4.