A mixed integer program to model spatial wildfire behavior and suppression placement decisions
Erin J. Belval, Yu Wei, and Michael Bevers

Abstract: Wildfire suppression combines multiple objectives and dynamic fire behavior to form a complex problem for decision makers. This paper presents a mixed integer program designed to explore integrating spatial fire behavior and suppression placement decisions into a mathematical programming framework. Fire behavior and suppression placement decisions are modeled using nodes associated with cell centers from raster landscapes. The nodes at which suppression is located are determined by control variables. Response variables include fire spread paths, arrival times, and fireline intensities for each node. Both fire arrival times and fireline intensities are necessary to address ecological objectives and fire control. Test cases for this model provide examples of fire behavior interacting with suppression placement to achieve multiple objectives.

Key words: wildfire growth, wildfire suppression, optimization, fireline intensity.


Mots-clés: croissance des feux de forêt, suppression des feux de forêt, optimisation, intensité de la ligne de feu.

1. Introduction
Over the past decade, wildland fire suppression costs have more than doubled in the United States (US). Appropriations from US Congress for wildfire suppression averaged 2.9 billion dollars annually from 2001 to 2007, compared with 1.2 billion from 1996 to 2000 (U.S. Government Accountability Office (US GAO) 2009a). These cost increases led the US GAO to call repeatedly for the U.S. Department of Agriculture (USDA) Forest Service (hereafter Forest Service) to develop cost-effective firefighting strategies (US GAO 2009a). The GAO criticized the five federal firefighting agencies (the Forest Service and four agencies from the U.S. Department of Interior: the Bureau of Land Management, the National Park Service, the Fish and Wildlife Service, and the Bureau of Indian Affairs) in 2007, stating “that officials in the field have few incentives to consider cost containment … and that … the lack of a clear measure to evaluate the benefits and costs of alternative firefighting strategies fundamentally hindered the agencies’ abilities to provide effective oversight” (US GAO 2009b). In 2009, the agencies still had not adopted any measures to evaluate alternative firefighting strategies (US GAO 2009b).

Cost is only one of many issues that forest managers must consider when fighting wildfires. Environmental values to consider include providing recreational opportunities, production of timber, protection of wilderness and wildlife, and providing high-quality water (US GAO 2009a). Other values at risk in a fire can include houses, infrastructure, high-value cultural sites, and even lives of both the firefighters and the public. Not only are many values at risk, but fire managers must also prioritize these values, which has been shown to be a complex and difficult task (Calkin et al. 2013). Wildfire suppression decisions combine multiple objectives and risk factors to form a complex background against which decision makers attempt to determine an efficient set of safe and effective management actions in a short period of time.

Mathematical methods have been developed in the past to address various aspects of fire suppression. For example, several mathematical programs have been built to examine fire containment dispatching decisions, i.e., how to assign resources to fires most efficiently. Many of these mathematical programs are quasispatial; they include spatial characteristics of the fire (for example, length of fire perimeter and (or) amount of area burned) rather than explicitly defining spatial fire behavior. Parks (1964) built one of the first models to determine the optimal number of firefighters to dispatch on individual fires. Inputs to Parks’ model were length of fire perimeter, amount of area burned, observed rates of fire growth, fireline production rate per firefighter, and costs. Parks’ model reduced the fire perimeter growth rate as a function of fireline production. Wiitala (1999) also addressed optimal dispatch for initial attack. He built a quasispatial, dynamic programming model based on exogenously determined functions of fire perimeter and area growth unaffected by fireline construction. Donovan and Rideout (2003) built a deterministic integer programming model to determine the optimal set...
of firefighting resources to dispatch to a fire for initial attack, while minimizing resource costs and fire damage. Their model also required exogenously determined lengths of fire perimeter and amounts of area burned but over discrete time intervals. Kirsch and Rideout (2003) extended Donovan and Rideout’s (2003) mixed integer programming model by including multiple fire ignitions. Another extension of the Donovan and Rideout (2003) model is presented by Hu and Ntiamo (2009), where a stochastic integer model is integrated into a simulation-optimization framework to determine firefighting resource dispatch plans. This stochastic optimization model is built on scenarios, each of which represents a fire day. For each fire day, the number of fires during the day, the length of the perimeter of each fire, and the amount of area burned per fire are stochastic. Explicitly spatial information is included in the model both before optimization (to determine fire perimeter lengths and amount of area burned) and after optimization (to simulate fireline construction); however, fireline construction was not optimized.

Deployment decisions, i.e., determining where suppression resources should be stationed, have also been examined using mathematical models. Haight and Fried (2007) developed a stochastic programming model to determine where to station fire engines within a single fire planning unit using stochastic fire-day scenarios and standard responses predetermined from CFES2 simulation runs (described below). They examined trade-offs between fire engine availability and the number of fires that did not receive a standard response. An extension of Haight and Fried’s (2007) model is presented in Lee et al. (2013) and allows the model to accommodate multiple fire planning units and multiple resource types. A similar model was developed by Ntiamo et al. (2012) that uses a two-stage, stochastic-programming, standard-response framework to determine the optimal configuration of resources at operational bases in the first stage and the optimal assignment of resources to fires in the second stage. The model uses scenarios and stochastic parameters similar to the model presented by Hu and Ntiamo (2009).

Other mathematical approaches have been used to model fire containment while accounting for the interaction between fireline production and fire growth. Fried and Fried (1996) developed a simulation algorithm in which fire growth is hampered by fireline production prior to fire containment. Their model has been used in several operational programs including CFES/CFES2 (a fire asset deployment model for California) and FPA (a fire preparedness budgeting model for the US) (Fried and Fried 2010). The model developed by Fried and Fried (1996) accounts for the interaction between elliptical fire growth and fireline production. HomChaudhuri et al. (2010) developed a simulation-optimization framework to place anchor points and quadratically shaped fireline segments to connect the anchor points on a landscape. Their model used a genetic algorithm to optimize fireline segment placements based on stochastic wind speeds and directions.

Some mathematical programs have been constructed to incorporate explicitly spatial fire behavior and to address suppression placement. Hof et al. (2000) built a mixed integer program (MIP) to optimize fire management decisions to slow fire that is headed towards a high-value area of land. Using a gridded landscape, the objective of the program is to maximize the amount of time before fire reaches the protected area. The fire arrival time constraints calculate arrival time by working counter to the objective function; the objective is to delay fire spread and the constraints force the fire to arrive at each cell at the earliest possible time. Another example of fire arrival times embedded in a mixed integer program is presented by Wei et al. (2011) and is used to determine suppression placement that minimizes the values lost due to a fire. Their fire arrival time constraints were based on the same assumptions as Hof et al. (2000). The optimization program was run twice, first to get optimal suppression placement and second to correct any nonbinding arrival times. The models built by Hof et al. (2000) and Wei et al. (2011) do include interactions between fire growth and suppression placement; however, fire arrival time is the only fire attribute calculated in their models and beneficial fires cannot be modeled. For example, in Wei et al. (2011), fire flame length is treated as a parameter in each raster cell and is precalculated without tracking the actual fire spread direction, which may be influenced by suppression placement.

Fireline intensity is a fire behavior characteristic that informs managers as to how much heat a fire releases per unit length of fire perimeter per unit time (Byram 1959). This characteristic could be used as a proxy for several important factors in fire management. For example, fireline intensity has been related to the likelihood of fire crossing a fireline (Hirsch et al. 1998), how long it takes to build a section of fireline (Holmes and Calkin 2013), firefighter safety (Butler and Cohen 1998), and ecosystem response to fire (Hood et al. 2007). Thus, managing the fireline intensity as a response to suppression activities can expand the accuracy, realism, and possible uses of a mathematical program that models fire behavior and related fire losses and benefits.

This paper presents a new mixed integer linear program (MILP) that models spatial fire behavior interacting with suppression placement. The model presented here calculates both fire arrival times and fireline intensities based on the direction that a fire spreads into a cell as a response to spatially explicit suppression placement. Test cases are examined using maps of the fire arrival times, spread directions, fireline intensities at each cell, and suppression locations. The model’s ability to determine efficient suppression placement decisions is demonstrated using test cases that examine trade-offs between suppression cost and area burned. Lastly, a test case is presented that examines suppression placement strategies for obtaining ecological objectives by lowering fireline intensities.

2. Methods

This MILP model contains four categories of constraints: spatial relationships, fire arrival time calculations, fireline intensity calculations, and suppression restrictions. Spatial relationship constraints determine how fire spreads between nodes. For this study, nodes are defined as points placed at the center of each cell on a rasterized landscape. We will refer to “nodes” when discussing properties assigned to the point at the center of the cell and to “cells” when discussing landscape properties that apply to the whole cell. We will refer to “fire spread paths” when discussing the links between nodes that allow fire to spread between them. Figure 1 shows a set of nine cells and their corresponding nodes, along with the set of fire spread paths into the center node. Fire arrival time constraints report one arrival time at each node on the landscape corresponding to a “binding fire spread path”. Fireline intensity constraints determine the fireline intensity along the binding fire spread path and classify the fireline intensity at each node as above or below a user-defined fireline intensity threshold. The fireline intensity constraints become particularly important when management concerns such as safety, line quality, and ecological objectives (presented in this paper) are incorporated into the model.

The fire spread paths into each node are response variables controlled by both the spatial relationship constraints and the fire arrival time constraints. Fire arrival times at each node and fireline intensities along each fire spread path are response variables determined by the fastest fire spread paths. The fire spread rate parameters are determined using constant weather. The assumption of constant weather will be addressed in the discussion and relaxed in studies that build on the model presented in this paper. We model suppression using “control locations” as decision variables that alter fire spread paths. If a control is located at a flammable node, we assume fire will not spread into that node.
Nonflammable nodes are predetermined during the model parameterization process, and fire will not spread into a nonflammable node. Control locations are incorporated into the model decisions to test fire behavior response to potential suppression activities. We plan to explore more realistic fire suppression decision variables in subsequent studies. An optimal solution to the mathematical program will report the set of selected control locations and resulting fire arrival times, the binding spread paths, and the fireline intensities along all fire spread paths.

Notation
The notation used to present the model is provided in the list of symbols. In this paper, uppercase Arabic letters are used to represent decision variables, lowercase Arabic and Greek letters indicate parameters, and uppercase Greek letters represent sets. An exception to this is “big M”, which is commonly used in if-then logic constraints.

Constraints

Spatial relationship constraints
To parameterize this model, landscape data are discretized into homogeneous square cells. A node is located at the center of each cell, and fire can spread from node to node based on the spatial relationships between cells (see Fig. 1). In this study, fire spreads only between nodes in adjoining cells (i.e., cells that share an edge or a vertex) as in NtaiMO et al. (2004) and Alexandridis et al. (2011). This work is based on raster data because much of the spatial data used in fire management is in raster form; however, the model itself is not dependent on using square raster cells, and a landscape can be divided into cells shaped as hexagons or triangles. The area within each cell is assumed to be homogeneous with respect to fuels, topography, and weather. However, fire spread rate and intensity are determined by the binding fire spread path that produces the earliest arrival time.

Equation 1 allows each node \( i \) (where \( i \) indexes all the nodes in the landscape) to burn only if the corresponding cell is flammable. If a cell is flammable and no control is located therein, then if one of its neighboring nodes burns or if there is an ignition within the node, the node itself must also burn (eq. 2). Fires may spread very slowly and may arrive at a node at any time but are only stopped by controls, nonflammable cells, or nonflammable landscape boundaries. Fires may ignite from multiple nodes, and thus a modeled fire can start at any size. Equation 3 ensures that if a node burns, exactly one spread path into it is identified; the model only tracks the first time that fire reaches a node (described later) and the corresponding binding spread path. Equation 4 allows fire to spread out of a node only if that node has burned. Equation 5 keeps nonflammable nodes and nodes protected with controls from burning; if fire is always assumed harmful, this equation is not needed.

Fire arrival time constraints
We only track the first time that fire arrives at a node. This is consistent with many previously developed fire spread algorithms, including Hof et al. (2000), Finney (2002), Alexandridis et al. (2011), and Wei et al. (2011). The fire arrival time constraints presented here assume that the earliest fire arrival time determines the binding spread path into a node from its neighbor; this is the only spread path recorded for the fire.

The fireline intensity for each node is determined not only by the fuel, terrain, and weather in each cell, but also by the direction in which the fire is moving. Equation 9 calculates the fireline intensity along the binding fire spread path into the node as determined by eq. 8.

Modeling fireline intensity in a spatially dynamic fashion is an important contribution of this model. For example, fire policy in the US requires fire managers to consider multiple objectives (Calkin et al. 2011). In areas that benefit from being burned, the fire might be monitored or managed at a lower intensity level but not actively suppressed, whereas in areas where fire threatens structures or other values at risk, the fire might be actively sup-

\[
\begin{align*}
(1) & \quad D_i \leq 1 - \xi_i \quad \forall i \\
(2) & \quad D_i + Y_i + \xi_i \geq \zeta_i + \frac{1}{n} \sum_{j \in \Omega_i} b_{ij} \quad \forall i \\
(3) & \quad \sum_{j \in \Omega_i} b_{ij} = D_i - \xi_i \quad \forall i \\
(4) & \quad b_{ij} \leq D_i \quad \forall i, j \in \Omega_i \\
(5) & \quad D_i \leq 1 - Y_i - \xi_i \quad \forall i \\
(6) & \quad F_i = f_i \quad \forall i | \xi_i = 1 \\
(7) & \quad F_i \leq F_j + b_{ji} + M(1 - D_j) \quad \forall i, j \in \Omega_i \\
(8) & \quad F_i \geq F_j + b_{ji} - M(1 - B_{ji}) \quad \forall i, j \in \Omega_i \\
(9) & \quad I_i = \sum_j \kappa_j b_{ij} + \kappa \xi_i \quad \forall i
\end{align*}
\]
pressed regardless of its intensity. This model reflects these different policy objectives by introducing a fireline intensity threshold \((g_i)\) that allows the model to determine nodes where fire is beneficial and where it is harmful. Each node on the landscape is preassigned a threshold that determines the fireline intensity at which fire becomes harmful for that location. For cells containing resources that are threatened by any fire (e.g., cells that contain structures), the threshold for that node can be set to zero, indicating that any fire in that cell is undesirable. For cells that could benefit from low-intensity fire, the threshold can be set to reflect the maximum desirable fireline intensity.

\[
\begin{align*}
(10) & \quad G_i + H_i \leq D_i \quad \forall i \\
(11) & \quad I_i \leq g_i + M|H_i + (1 - D_i)| \quad \forall i \\
(12) & \quad I_i \geq g_i - M|G_i + (1 - D_i)| \quad \forall i
\end{align*}
\]

Equation 10 ensures that if a node burns, then the fire in it is classified as either beneficial or harmful. When the fireline intensity is above the threshold, fire must be classified as harmful to make eq. 11 feasible. Similarly, when the fireline intensity is below the threshold, fire must be classified as beneficial to make eq. 12 feasible. If the fireline intensity is exactly equal to the threshold, the model will choose whichever classification improves the objective function. The threshold allows suppression actions to be based on the goal of managing fire behavior rather than solely on containing the fire.

**Suppression restrictions**

We use control locations as a simple mechanism to test how suppression could be incorporated into the model to influence fire behavior. If controls are spatially unconstrained and all fire is considered harmful, a control always would be placed in the ignition node to stop the fire before it could spread, making the placement decisions trivial. To avoid this and to reflect situations such as delayed arrival of suppression resources, in each test case, we identify in eq. 13 a set of nodes where controls may not be placed.

\[
\forall i \in \Xi
\]

**Objective function**

By design, the constraints governing fire arrival time and fireline intensity allow for flexibility in the objective function by accounting for both beneficial and harmful effects of fire. The ability to employ objective functions reflecting various priorities is another important attribute of the model. Examining different weighting schemes for multiple objectives may help managers determine how different priorities might translate into alternative suppression strategies.

We used the two following objective functions for the test cases presented in this paper:

\[
\text{minimize } Z = \sum_i w_Y Y_i + \sum_i w_D D_i
\]

\[
\text{minimize } Z = \sum_i w_Y Y_i + \sum_i w_H H_i - \sum_i w_C C_i
\]

The objective function (Z) in eq. 14 minimizes the value of the area burned and the number of nodes at which a control is located, weighted to reflect control costs and the value to be protected from fire in each cell. This objective function assumes that all fire on the landscape is detrimental. Equation 15 includes ecologically beneficial fire in the objective function by minimizing the value of cells burned above the intensity threshold plus control costs while maximizing the value of cells that burn below the intensity threshold, again using weights to reflect the losses and benefits associated with the area burned and the cost associated with controls.

**Test cases**

In this paper, we present three sets of test cases. In the first set, we run the model on two different landscapes without controls \((\Xi\) in eq. 13 is the set of all nodes). We used eq. 14 as the objective function for the runs in this test case; however, with no controls, the objective function has no impact on the solution. In the second set of test cases, we examine trade-offs between control costs and area burned, again using eq. 14 as the objective function. For these model runs, we constrained the controls so that they could not be placed in any node with a fire arrival time prior to a predefined threshold. Such constraints could be used to reflect resource arrival time. To create these constraints, we first ran the model with no controls to identify the set of nodes (i.e., \(\Xi\) where fire arrived before 40 min. Then we resolved the model, allowing controls to be placed in all cells except those members of \(\Xi\). The third test case is a single model run examining how fire behavior can be altered rather than just suppressed, using eq. 15 to incorporate both fire benefits and losses into the objective function. In this test case, controls could be placed at any node in the landscape; therefore, the set of nodes where controls could not be located was empty.

The initial result that we present in the first set of test cases uses a 6 cell \(\times\) 6 cell homogeneous landscape. This landscape is flat and is composed of cells containing fuel type 2 (Anderson 1982), which models a “timber-grass understory” in which fire is carried mainly through the herbaceous fine fuels. This simplified landscape is big enough to show a full set of model solutions and small enough to show the details of the model output using figures. All other test runs in this paper use an 11 cell \(\times\) 12 cell heterogeneous landscape modeled from a location in the Black Hills of South Dakota (see Fig. 2). Table 1 summarizes the characteristics for each cell in the 6 \(\times\) 6 homogeneous landscape and the range of values for the 11 \(\times\) 12 heterogeneous landscape. The 11 \(\times\) 12 heterogeneous landscape has some gentle to moderate slopes, gaining 32 m in elevation over 330 m distance predominately with a northerly aspect, and is composed of the following fuel types: 102, a “low load, dry climate grass” fuel type where fire is carried by grass; 122, a “moderate load, dry climate grass–shrub” fuel type where fire is carried by grasses and shrubs; and 188, a “long-needle timber” fuel type where fire is carried mainly by needle litter (Scott and Burgan 2005). We designated 13 cells to be nonflammable for demonstration purposes. All landscape boundaries are assumed to be nonflammable. The 11 \(\times\) 12 landscape covers an area of approximately 0.12 km\(^2\) (11.88 ha). Wind is assumed to be from the south. For all test cases, the fireline intensity threshold was set to 100 kW\(\cdot\)m\(^{-1}\) (about 0.65 m flame length) to indicate if fire is beneficial (cells burned under 100 kW\(\cdot\)m\(^{-1}\)) or harmful (cells burned at a fireline intensity over 100 kW\(\cdot\)m\(^{-1}\)). This fireline intensity threshold affects only the results in the third test case, which uses eq. 15 for the objective function.

FlamMap was used to calculate the fire spread rates for every cell based on Rothermel’s equation and the maximum fireline intensity based on Byram’s fireline intensity (Finney 2006). FlamMap outputs include the major fire spread direction in each cell, the parameters of an ellipse that describes the within-cell fire shape, and the maximum fireline intensity. The rates of spread between a raster node and its neighbors can be calculated using standard elliptical spread assumptions (Wei et al. 2011). Fireline intensity is often calculated as \(I = H_{or}t\), where \(I\) is the fireline intensity (also called Byram’s fireline intensity), \(H\) is the fuel heat of combustion, \(\omega\) is the weight of fuel consumed per unit area in the active flaming zone, and \(r\) is the rate of spread (Byram 1959). FlamMap only reports the fireline intensity along the direction of

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maximum rate of spread. Given fire spreading from node \( j \) to neighboring node \( i \), we approximated fireline intensity in the direction of interest as follows:

\[
\kappa_{ji} = \frac{1}{2} \left( \frac{I_i}{R_{\text{ROS}i}} + \frac{I_j}{R_{\text{ROS}j}} \right)
\]

where \( I_i \) is the fireline intensity along the direction of maximum rate of spread within node \( i \), \( \kappa_{ji} \) is the intensity of fire in node \( i \) if fire spreads from node \( j \) to \( i \), \( R_{\text{ROS}i} \) is maximum rate of spread in node \( i \), and \( R_{\text{ROS}j} \) is actual rate of spread from node \( j \) to node \( i \) within node \( i \). If an ignition is placed in node \( i \), then the fireline intensity in that node is approximated as the fireline intensity corresponding to the maximum rate of spread.

\[ \kappa_i = I_i \]

Visual Basic and CPLEX 12.6 were used to parameterize and solve the models using the solver’s default environmental settings. All but one test model solved to optimality in less than 10 s on a computer with 3 GB of available RAM. We discuss scalability and solution times in the Discussion.

### 3. Results

#### Test case 1: examining fire behavior without controls

The results from running the model without controls on a flat, homogeneous, \( 6 \times 6 \) landscape are summarized in Fig. 3 with two separate maps. All of the maps use the convention that north is the top of the map. Equation 14 was used as the objective function, but without controls, the results are unaffected by the objective function. Figure 3a shows the fire arrival times for each node. Figure 3b shows the binding fire spread paths and fireline intensities; each arrow indicates the fire spread path into each node, and the grayscale shows the fireline intensity at which fire spreads into each node. Because of homogeneity across this landscape and no slope, the fireline intensity changes depending on the degree to which the spread path is aligned with the wind. For example, fireline intensity at node (C3, R1) is high because fire spread from node (C3, R2) to node (C3, R1) was exactly aligned with the wind. In contrast, the fireline intensity at node (C3, R5) is low because the fire spread path into that node was directly against the wind.

The \( 6 \times 6 \) landscape is completely flat and homogeneous. Therefore, given that the wind is coming from the south, we expect to see symmetrical spread paths into (C2, R1) and (C4, R1). However, Fig. 3b shows a solution with asymmetrical spread paths, i.e., fire spreads vertically from (C2, R2) to (C2, R1) and diagonally from (C3, R2) to (C4, R1). This asymmetry is an artifact of the homogeneous landscape: the spread path is bound to be the fastest spread, but there are two spread paths that give the same arrival time for both (C2, R1) and (C4, R1). For (C2, R1), fire spreading from either (C2, R2) or (C3, R2) will arrive at exactly the same time. Similarly for (C4, R1), fire spreading from (C3, R2) will arrive at exactly the same time as fire spreading from (C4, R2). Given fire spread paths that result in the same fire arrival time, the model is able to choose the spread path that results in a better solution or make an arbitrary choice when either is equally good. For example, if nodes burned at low fireline intensities are treated as beneficial in the objective function, the model would choose fire spread paths that increase the number of low-intensity nodes in concert with other objectives. Ties like this are unlikely to appear in heterogeneous landscapes, as even the smallest difference between cells will affect the arrival time associated with each spread path.

For all remaining test cases presented in this paper, we use the heterogeneous \( 11 \times 12 \) landscape shown in Fig. 2 and described in Table 1. Figure 4 shows the results of a run without controls, again using eq. 14 as the objective function. The results in Fig. 4 demonstrate a fire spreading around nonflammable barriers. The major driver of fireline intensity is fuel type. For example, cells (C1, R1–R4), (C2, R3–R6), (C3, R4–R7), and (C5, R5–R8) are all fuel type 102 (low load, grass) and all have fireline intensities under the fireline intensity threshold. Cells with heavier fuel loads (fuel types 122 and 188), e.g., cells (C2–C7, R1), (C2–C8, R2), and (C5–C9, R3), have higher fireline intensities. Cases in which fuel type does not drive the fireline intensity include where the fire is moving against the wind as in cell (C3, R11) or is moving down a steeper slope as in cell (C10, R1).

#### Test case 2: examining trade-offs between area burned and control costs

Figure 5 shows a parametric analysis on control costs \( w_C \) (Dykstra 1984). For this analysis, we used the objective function in eq. 14 with \( w_C = 1 \), which assumes that fire is always harmful. We varied \( w_C \) to determine the thresholds at which the control locations changed. Fire control locations were restricted such that they could not be implemented until 40 min after the fire started (eq. 13). In all four results shown in Fig. 5, the model links control locations with nonflammable nodes or the nonflammable landscape boundary to efficiently contain the fire. For the results in Fig. 5a, controls are relatively inexpensive, i.e., the cost of control

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Homogeneous 6 × 6</th>
<th>Heterogeneous 11 × 12</th>
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</thead>
<tbody>
<tr>
<td>Cell size</td>
<td>30 m × 30 m</td>
<td>30 m × 30 m</td>
</tr>
<tr>
<td>Fuel model</td>
<td>2</td>
<td>102, 122, 188</td>
</tr>
<tr>
<td>Elevation</td>
<td>1380 m</td>
<td>1357–1397 m</td>
</tr>
<tr>
<td>Slope</td>
<td>0%</td>
<td>0%–19%</td>
</tr>
<tr>
<td>Aspect</td>
<td>Flat</td>
<td>Generally northern</td>
</tr>
<tr>
<td>Canopy cover</td>
<td>0%</td>
<td>Varies with fuel type</td>
</tr>
<tr>
<td>Fuel moisture (1 h, 10 h, 100 h, live)</td>
<td>11, 20, 26, 100</td>
<td>11, 20, 26, 100</td>
</tr>
<tr>
<td>Wind speed</td>
<td>3.58 m s⁻¹</td>
<td>3.58 m s⁻¹</td>
</tr>
<tr>
<td>Wind direction</td>
<td>Southerly</td>
<td>Southerly</td>
</tr>
</tbody>
</table>

Table 1. The landscape values for each cell in the homogeneous 6 cell × 6 cell test case and the range of values for the heterogeneous 11 cell × 12 cell landscape.
at a node is less than the value lost if the node burns. The results suggest allocating controls at 18 nodes to keep fire from reaching any node that has not burned before the 40-min time mark. As controls become more expensive, the locations of controls change. For example, when control at a node costs the same as burning a node, then controls are removed from nodes (C6, R4) and (C9, R6). This allows fire to burn both nodes without spreading further and reduces controls from 18 nodes to 16 nodes (Fig. 5b). At this threshold, four alternative optimal solutions occur; the objective function value stays the same regardless of whether controls are placed at nodes (C6, R4) and (C9, R6) or if they burn. Two of the four optimal solutions are shown in Figs. 5a and 5b. When control at one node is worth more than the value of 2.5 nodes being burned, another threshold occurs. The pattern of control changes again, allowing more area to burn (i.e., nodes (C8, R3), (C9, R3), (C8, R4), (C9, R4), and (C9, R5) burn) while locating controls at only 14 nodes (Fig. 5c). Similar to the previous threshold, when one node of control is worth exactly the value of 2.5 nodes being burned, alternative optimal solutions (two at this threshold) again occur; the objective function value stays the same for the solutions in Figs. 5b and 5c. Figure 5d shows that once a node of control is worth 3.35 times the value of a node burned, the optimal solution connects the nonflammable nodes in the center of the landscape to the northern landscape boundary. Eventually, controls become expensive enough ($w_Y = 7.8$) that it is no longer efficient to put any controls on the fire. When the ratio of $w_Y$ at one node to the value lost from burning one node is almost eight to one, then the optimal solution allows the whole landscape to burn. In this solution, the model is implicitly taking advantage of the nonflammable border of this landscape.

**Test case 3: accounting for beneficial fires**

Figure 6 shows the results of utilizing the objective function in eq. 15. This test case demonstrates how the model could be used to...
manage trade-offs between suppression costs, beneficial fire, and detrimental fire. In this scenario, the goal is to burn as much area as possible under the 100 kW·m⁻¹ fireline intensity threshold while minimizing the area burned above that fireline intensity. Fire control is available, and the costs of control are small but present. In this example, we assume that the value gained from burning five nodes below the intensity threshold offsets the value lost from burning one node above the threshold \( w_Y = 1 \) and \( w_H = 5 \). The cost of locating control in one node is equal to 10% of the value gained if fire burns one node below the fireline intensity threshold \( w_C = 0.1 \). Controls may be placed in any node on the landscape. Comparing this result with the “without controls” result shown in Fig. 4, it is clear that some of the choices in control locations are not just to keep nodes from burning, but rather to change the binding fire spread paths so that the fire burns more nodes at a beneficial (lower) intensity. All of the nodes surrounding the ignition \( (C7, R6) \) would burn at a harmful fireline intensity, but for any node on the landscape to burn at a beneficial intensity, at least one of the nodes surrounding the ignition must be allowed to burn. Therefore, node \( (C6, R7) \) is allowed to burn at a harmful fireline intensity, but other nodes that burn at a beneficial intensity outweigh the harm in that node (according to this objective function). Although node \( (C2, R1) \) is allowed to burn at a detrimental intensity, allowing fire to go through that node combined with optimal controls changes the fireline intensity of 12 nodes to the east from detrimental to beneficial. Similarly, node \( (C11, R5) \) burns at a detrimental fireline intensity to allow five nodes to the south to change from detrimental to beneficial fireline intensity. Controls are placed to direct the fire so that the fireline intensity is beneficial in nearly all of the burned nodes. These results demonstrate that even with limited control options, the model effectively uses controls to change fire behavior in a more sophisticated manner than simply stopping the fire.

The results shown in Fig. 6 took 40 h to run to optimality on a computer with 32 GB of available RAM. This is a substantially longer computation time than any of the other runs shown in this paper, all of which ran to optimality in less than 10 s. Interestingly, a 20-min run for this test case produced a solution that is less than 1% worse than the optimal solution found after 40 h of computation, although after 20 min, CPLEX still indicated a 21% gap. Running times and scaling issues are further addressed in the Discussion.

**Discussion**

The mixed integer programming model presented in this paper provides a new framework to model spatially dynamic fire behavior in a mathematical program. The fire behavior modeled includes fire arrival times, fireline intensities, and binding fire...
spread paths. The ability to model controls that interact with fire behavior is novel and allows the model to examine fire management objectives that are more complex than just containment. The interactions between controls and fire behavior are highlighted by the test cases that examine trade-offs between suppression cost and the value of area burned, where the pattern of control changed in response to an increased suppression cost. Managing the spatially dynamic nature of fire in an ecological context is also possible, because the model includes constraints that characterize fire as beneficial or harmful based on fireline intensity. These constraints allow the model to employ different management objectives, including minimizing total fire losses and balancing fire loss with benefits. The test case examining fire benefits demonstrates how the interactions between suppression and fire behavior can be influenced by management objectives, as control locations were selected to lower fireline intensity rather than to simply contain the fire or minimize the cost. Although our model classifies individual nodes and associated cells as burning beneficially or harmfully, many landscape-level objective functions can be constructed from this type of information by accounting for spatial relationships among cells (Hof and Bevers 2002).

This initial model formulation is deterministic if weather is kept constant. If the planning horizon includes weather changes, the model could be run in an iterative fashion to use different sets of predicted weather parameters (e.g., Finney 2004, 2006). However, crafting a single solution from multiple runs poses a major challenge and likely would lead to suboptimal decisions. Given how dramatically weather changes can affect fire behavior, this is an important limitation of this model. In addition, fire containment requires decisions to be made, whereas future weather is still uncertain. Thus, solutions to deterministic models may not be robust over the set of possible future weather scenarios and may not reflect the typically risk-adverse attitudes of fire managers. In following work, we plan to adapt this model to accommodate multiple weather streams that will allow stochastic weather changes to drive decisions.

Our model currently does not incorporate a number of important fire behaviors, including spotfires and reburn. Spotting can drive fire spread and allow fire to jump over control lines. The model could be altered to allow for spotfires; however, the neighborhood that allows fire to spread from node to node would need to be expanded so that fire could spread between nodes with associated cells that do not share an edge or vertex. New decision variables representing ignitions by spotfires would be necessary. However, these changes would make the model larger. Also, adding spotting might further necessitate moving the model into a stochastic framework. Reburn, the phenomenon in which fire burns over an area more than once, is not included in this model either. We anticipate that reburn could be incorporated into this model by adding a second set of fire behavior decision variables for each cell, i.e., a second network of fire spread that would be activated only after the fire had reached a cell the first time. Our model is consistent with current operational models (Finney 2004, 2006) in only reporting one set of fire behaviors for each cell. Extending the model to incorporate reburn would significantly increase the size of the model. Another fire behavior assumption used in this model is that fireline intensity is only dependent on which node $j$ directly spreads fire to node $i$, not on the entire path from the ignition to the node. Similarly, we only account for the fire behavior resulting from the earliest arrival time; we do not account for the effects of fire arriving at nearly the same time from different directions. Although these assumptions are also consistent with current operational models of fire behavior (Finney 2004, 2006), they can make the scaling of the raster cells important. For example, smaller raster cells allow for finer scale, more realistic fire behavior; however, these cells are more likely to have their fire behavior influenced by cells beyond their directly adjacent neighbors.

An additional limitation is that the fire control decisions that this model supports are simplistic. Although modeling the location of controls does allow the fire to interact with suppression decisions for testing purposes, it does not take into account many important facets of suppression. For example, control locations can only represent the location of suppression actions; they do not account for the temporal dynamics of control activities. We used a predefined resource arrival time in test case 2 to address resource travel time, but this method does not address issues such as the time required to implement controls. As implemented here, the controls are highly simplified and cannot model a number of suppression activities, including burnout operations and aerial firefighting. Also, modeling suppression using these simplified controls assumes that control is always effective. In reality, if the fireline is not robust enough, fire may jump the line. Similarly, the pattern of control locations selected by this model may be robust over the set of possible future weather scenarios and still uncertain. Thus, solutions to deterministic models may not account for the fire behavior resulting from the earliest arrival time; we do not account for the effects of fire arriving at nearly the same time from different directions. Although these assumptions are also consistent with current operational models of fire behavior (Finney 2004, 2006), they can make the scaling of the raster cells important. For example, smaller raster cells allow for finer scale, more realistic fire behavior; however, these cells are more likely to have their fire behavior influenced by cells beyond their directly adjacent neighbors.

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be difficult to implement. For example, isolated suppression activities on the interior of a fire (e.g., Fig. 5a, node (R4, C6)) may be undesirable. Control locations as implemented in this model provide a demonstration of the response of fire spread and intensity to suppression activities but cannot address the issues above. In future work, we plan to adapt this model to address control timing, line quality issues, and firefighter safety.

We discovered that selecting different objective functions can influence computer solution times for this model. Objective functions solely focused on minimizing losses (i.e., eq. 14) tended to run substantially faster than those that consider both the beneficial and detrimental effects of fire (i.e., eq. 15). We hypothesize that the difference in running time is due to the variation in model complexity. For example, fireline intensity at a location may change from detrimental to beneficial if the spreading path of the fire is altered by controls. However, the same controls may change fire paths such that fire becomes harmful elsewhere. Due to this added complexity, the multi-objective problem is more difficult to solve than the simpler containment-only problem.

Because this is an explicitly spatial model, as the landscape size increases, the model size increases geometrically. Although the 11 x 12 landscape tested does cover almost 12 ha, we anticipate that managers will want to examine larger landscapes. The solution algorithms that we used lead to very large initial MIP gaps, especially for objective functions incorporating both fire benefits and losses (as in eq. 15). Consequently, scaling up could prove difficult. The initial MIP gap for the results in Fig. 5 ranged from <1% to 85.38%. With the smaller problem, CPLEX is able to close this gap quickly; however, as the problem size increases, closing the gap takes longer. Using this model operationally on a larger landscape may require heuristic solution methods.

Despite the limitations identified above, the model presented in this paper is a step forward in modeling fire spread and suppression in a mathematical programming framework. The inclusion of fireline intensity in addition to fire arrival times provides many options for further extensions of this model. For example, future extensions of the model might incorporate fireline intensity into safety constraints while ensuring that the fireline produced is robust enough to withstand the arriving flame front. The deterministic model presented in this paper provides a basis for designing mathematical programming models that incorporate stochastic weather and more realistic suppression activities.

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References

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List of symbols

Decision variables

\( D_i \) binary indicator: \( D_i = 1 \) indicates that node \( i \) burned

\( Y_i \) binary indicator: \( Y_i = 1 \) indicates that a control has been placed at node \( i \)
Binary indicator: $B_i = 1$ indicates that the binding fire spread path (earliest arrival time) into node $i$ was from node $j \in \Omega_i$ (defined below).

Fire arrival time at node $i$  

Fireline intensity along the fire spread path into node $i$  

Binary indicator: $G_i = 1$ indicates that node $i$ burned below a predefined fireline intensity threshold.

Binary indicator: $H_i = 1$ indicates that node $i$ burned above a predefined fireline intensity threshold.

**Parameters**

- $n_i$: number of nodes adjacent to node $i$
- $b_{ji}$: amount of time it takes fire to spread from node $j$ to node $i$
- $\kappa_j$: fireline intensity of the fire spread path from node $j$ to $i$
- $\kappa_i$: fireline intensity of the fire in node $i$ if the fire ignited in node $i$
- $\xi_i$: binary indicator: $\xi_i = 1$ indicates that fire ignited from node $i$

**Sets**

- $\Omega_i$: the set of nodes adjacent to node $i$
- $\Xi$: a set of nodes that cannot have controls placed within them for the modeled fire, i.e., due to the delayed arrival of suppression resources.

- $\ell_i$: binary indicator: $\ell_i = 1$ indicates that node $i$ is nonflammable.
- $f_i$: ignition time for fire starting from node $i$ (only available if $\xi_i = 1$)
- $M$: a relatively large number ("big M")
- $g_i$: predefined fireline intensity threshold for node $i$
- $w_{lo}$: the value lost if node $i$ burns
- $w_c$: the cost to put a control at node $i$
- $w_{va}$: the value gained by allowing node $i$ to burn at a beneficial fireline intensity
- $w_{hl}$: the value lost if node $i$ burns at a harmful intensity