

Consistent Definition and Application of Reineke's Stand Density Index in Silviculture and Stand Projection

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Abstract—Reineke's Stand Density Index (SDI) has been available to silviculturists for over 75 years, but application of this stand metric has been inconsistent. Originally described as a measurement of relative density in single-species, even-aged stands, it has since been generalized for use in uneven-aged stands and mixed-species stands. However, methods used to establish the maximum SDI for various forest types have varied widely. As a result, there are maximum SDI values for some forest types that do not appear to be supported by adequate analysis. This situation has led to confusion and lack of confidence in SDI among some practitioners. We describe several issues related to the determination of maximum SDI, and propose guidelines for future research and application.

Keywords: density management diagram, Forest Vegetation Simulator (FVS), Forest Inventory and Analysis (FIA), SDI, self-thinning, stand development

Introduction

Reineke's Stand Density index (SDI; Reineke 1933) is a useful measure of relative density that, in silvicultural practice today, forms the basis for most density management diagrams (Jack and Long 1996; Newton 1997). SDI is based on the relationship between tree size and number, and the changes in both as a stand develops and self-thins as it matures. As such, SDI and density management are grounded in the so-called $-3/2$ self thinning law, although the "law" itself was not well-described until the 1950s and 1960s by Japanese researchers (Kira and others 1953; Shinozaki and Kira 1956; Yoda and others 1963). In its original form, the law describes the inverse relationship between the average mass of plants in a population and their number (Zeide 1987):

$$w = kN^{-3/2} \quad [1]$$

where: w is average plant mass,
 N is the number of individuals per unit area,
and k is a constant, generally varying by species.

In common forestry application, including SDI, the quadratic mean diameter is substituted for average mass. Although the universality of the law was affirmed by subsequent investigation (White and Harper 1970), it has been a source of controversy among some researchers (Osawa and Sugita 1989; Pretzsch and Biber 2005; Sackville Hamilton and others 1995; Weller 1987; Weller 1990) and the discussion continues today.

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Although many of the issues surrounding the characteristics and use of SDI have been synthesized elsewhere (Shaw 2006), several issues are particularly relevant to the use of SDI in silviculture, stand characterization, and stand projection within the Forest Service. Some of these issues are as fundamental as a lack of agreement on the maximum SDI value that should be used for a particular species or forest type across different Forest Service Regions. Indeed, at this workshop Basford (this proceedings) described how the lack of a suitable maximum SDI led to adoption of a different, although related approach to density management in ponderosa pine forests of Idaho. Guidelines specified a maximum SDI of 830, when the appropriate maximum should not have exceeded 450 (Long and Shaw 2005).

The maximum SDI of 450, which is used in other parts of the Interior West, was supported by a range-wide analysis of ponderosa pine (Long and Shaw 2005). However, the 830 maximum used in Idaho should have been questioned much earlier because it can be traced back to Reineke's (1933) original paper. Close examination of Reineke's graph of ponderosa pine data reveals that the ponderosa pine maximum of 830 was based on fewer than two dozen data points, and in none of the cases did mean diameter exceed 6 inches. Surely, the persistence of this unsupportable value in Forest Service documentation suggests that it was never examined closely. Through experience, practitioners suspected there was a problem, but it was not caught by the research community. As a result, experiences such as Basford's have led to diminished trust in SDI as a useful index, in turn causing some to abandon its use. Because of controversy in the literature and perceived shortcomings in practical use, other practitioners have been hesitant to adopt the use of SDI or density management diagrams (fig. 1).

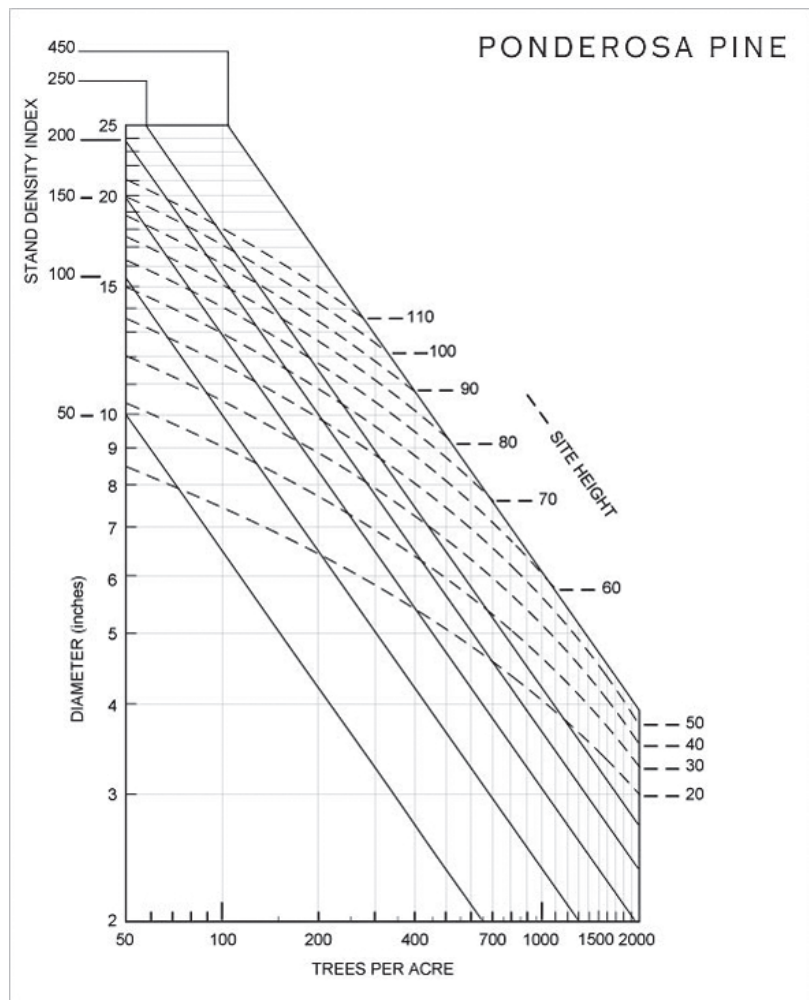


Figure 1—Density management diagram for ponderosa pine. Analysis of data from across the range of ponderosa pine in the western United States supported the use of 450 as the maximum SDI (Long and Shaw 2005). The slope of the maximum SDI line is based on the self-thinning relationship.

However, we have found that many of the “problems” with SDI may be traceable to inadequate data, inconsistent methods, or even the perpetuation of weakly supported results. From the perspective of the silviculturist, these problems represent a failure of the research and development community to address the needs of the practitioner—i.e., a lack of clarity in research leads to confusion in application. In this paper we highlight the most important issues and propose a path to resolution. The need for resolution is driven by the need for consistent definition and use of SDI within the Forest Service:

- Forest Service silviculturists need defensible measures of relative stand density to use as the basis for silvicultural prescriptions; these may be in the form of a maximum SDI for a given forest type or a well-constructed density management diagram. Research that relates certain stand characteristics, such as risk of insect infestation or suitability as wildlife habitat, to SDI, must measure stand density in a way that is consistent with the values used by the silviculturist.
- The Forest Vegetation Simulator (FVS) uses SDI as a driving variable during projections of stand development (Dixon 2003). FVS requires generalized rules for using SDI in models that predict stand response to changing composition, structure, and density. Values used by these models should be consistent, in terms of the data and analysis used, with the methods employed by model users—e.g., silviculturists.
- The Forest Inventory and Analysis (FIA) program has a need to assess and interpret stand characteristics, including relative density, in all forest types of the U.S. Regional inconsistencies in the characterization and use of SDI create difficulties with FIA analysis and reporting. The FIA program needs a sound basis for calculation of relative density.

Although some of these needs apply more broadly than to the Forest Service alone, it is important for the agency to strive for internal consistency among various tools and programs, such as FVS and FIA. Once this consistency is accomplished it will benefit the silviculture community at large.

The Issues

Some of the issues surrounding the use of SDI can be related back to one of the first questions that a silviculturist might ask in the development of a density management regime—“What is the appropriate maximum SDI to use for this stand?” This question requires a definition of the term “maximum SDI,” and implies some population to which it applies—i.e., the stand. The establishment of a maximum SDI for a particular forest also establishes the variable k for that type. In our experience, it is necessary to address four important issues in order to achieve consistent definition and application of SDI:

- Sampling methodology and error—stands at maximum SDI are rarely observed in the field, so it is necessary to understand the role of sampling in the estimation of maximum SDI.
- The scale issue—to what entity should a stated maximum SDI apply—the plot, the stand, or something else?
- Assumptions of universal slope and linearity—is the slope of the size-density relationship truly constant when plotted in log-log space, is it nonlinear, or are there other limitations to size-density?
- Lack of consistent definition—is there consistency between the data and methods used to determine maximum SDI and the application of the index in the field?

Sampling Error

Maximum SDI is generally understood to be a hypothetical limit of relative density that is very rarely observed in the field. However, it is common for data to include observations of higher SDI (sometimes much higher) than would be expected for well-studied forest types. These seemingly contradictory data are typically plot-based measurements that sample a relatively small portion of the population in question (i.e., the stand). One reasonable explanation for these “excessive” density observations could simply be that they represent a form of sampling error—i.e., the sample design captures a high-density portion of the stand but does not account for lower-density areas that affect the competitive status of the measured area. This possibility is related to the scale issue, which is discussed in more detail below.

Another possibility is that the methods used to establish maximum SDI for the forest type were different from the methods used to assess the stand at hand. In effect, the researcher who documents maximum SDI and the practitioner who assesses the stand are using different measurement scales. For example, consider a situation where the maximum SDI for a species was determined using plot-level measurements as the sample data. Although various methods may be used to establish the maximum, most methods allow for a small percentage of “outliers”—high-density plots that are considered to be the result of sampling error—and establish the maximum based on the remainder of observations or some subset.

In practice, however, density is assessed at the stand level. As a result, a potential problem arises because plot-level density and stand-level density are not equivalent measures. This can be illustrated with a simple example. Given an acre of forest (representative of a single stand) that is sampled by a series of four plots (fig. 2), SDI can be calculated five times—once for each plot and again at the stand level.

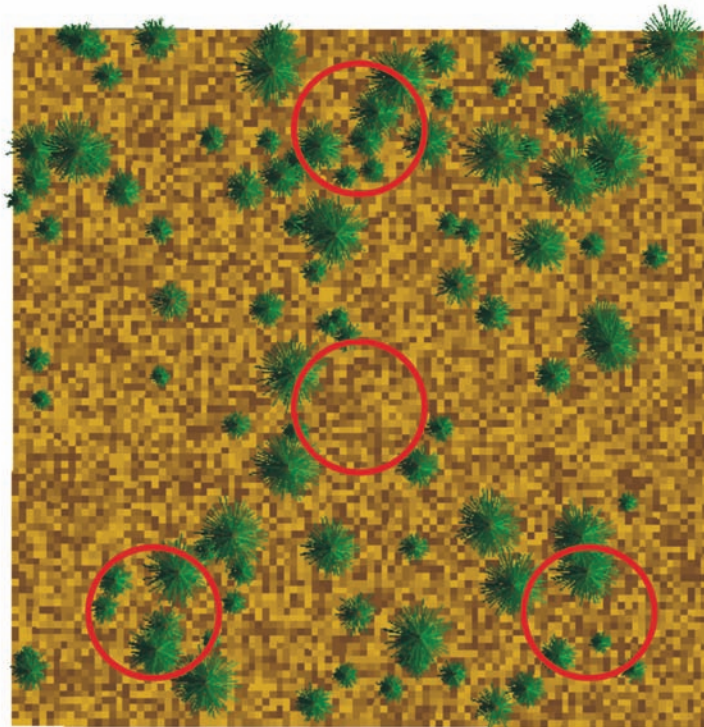


Figure 2—Example of within-stand variability that results in widely varying SDI at the plot level. SDI is 37 percent of maximum at the stand level, but the SDI of individual plots may be nearly double that.

Note that even in a relatively homogeneous stand, there is variability in SDI among the plots. In this example, SDI ranges from 0 to 245 among the plots, with a stand average of 137. Stand-level SDI is always lower than the maximum plot-level SDI, except in the special case that SDI is identical on all plots measured within a stand, because it is an average of the plot-level measurements. If we assume the example stand to be ponderosa pine, then plot-level SDI ranges from 0 percent to 54 percent of maximum SDI for the forest type, with stand-level SDI at just over 30 percent of maximum. From a silvicultural perspective, some parts of the stand have open growing space, some parts are on the verge of self-thinning (which is expected to begin at around 50% of maximum SDI), and the stand, on average, is relatively well-stocked. How then should the silviculturist assess this stand, when within-stand variation suggests a need for regeneration in some areas, thinning in others, and the average stand condition suggests that no intervention is needed? We suggest that the answer partly depends on how well-matched the assessment methods are to the methods used to determine the reference condition (i.e., maximum SDI). This situation changes the issue from one of sampling error (accurately measuring the condition) to one of plot scale (measuring the same thing in research and application).

The Scale Issue

When the scale issue is examined using a large sample of plots and stands, such as would be used to develop maximum SDI values, differences in the characteristics of the sample become apparent. Data from the Forest Inventory and Analysis (FIA) program are used to illustrate the effect of using plot-level vs stand-level data for analysis. The FIA plot design uses four 1/24-acre subplots, arranged as in figure 2. In most cases, all four subplots sample a single "condition," which is equivalent to a stand in most cases. Where multiple conditions (stands) occur across the plot footprint, the changes are mapped and the proportions of plot area occupied by each condition are recorded. For the purpose of this analysis, data from single-condition FIA plots were compiled two ways: 1) treating each subplot as a separate sample, using subplot data as the observation, and 2) averaging all subplots on a plot to represent a stand-level observation. The two estimates represent the exact same sampled area, with the plot-based data (1) having four times the number of observations as the stand-based data (2). The results of this comparison for two common forest types are shown in table 1.

Note that the measures of central tendency are relatively close for each forest type when the plot-level data are compared with stand-level data. However, the discrepancy between the two data sets increases substantially as the upper limits of apparent density are approached in each sample. The magnitude of this discrepancy is extremely important with respect to the estimation of maximum SDI, because

Table 1—Summary statistics for two forest types, using FIA data compiled at the plot level (FIA subplot) and at the stand level (FIA plot).

Sample Statistic for SDI	FIA Forest Type			
	Douglas-fir		Aspen	
	Stand-based	Plot-based	Stand-based	Plot-based
Mean	198	207	220	231
Median	191	186	211	207
Maximum	581	987	639	1683

all of the methods that have been used to estimate the maximum—from Reineke's pencil and ruler to RMA regression (Leduc 1987) to binning methods (Bi 2001; Bi and Turvey 1997)—manipulate the upper limits of the data in order to arrive at an estimate. As a result, it is possible for two studies to sample the same area of one population, apply the same analysis methods, but estimate substantially different maximum SDIs solely due to the sample design.

Assumptions of Universal Slope and Linearity

Another approach to SDI analysis that can be a source of inconsistency involves the determination of the slope of the self-thinning trajectory. As mentioned earlier, this has been a source of debate. However, this debate is more generally focused on the merits of the $-3/2$ self-thinning law as opposed to Reineke's SDI in particular (although the two are closely linked). The slope representing constant SDI is fixed by definition as reflected in the equation used to calculate it (equation 2):

$$SDI = TPA \cdot \left(\frac{Dq}{10} \right)^{1.6} \quad [2]$$

where SDI is stand density index,
 Dq is quadratic mean stand diameter in inches at breast height,
 TPA is the number of trees per acre,
 10 is the reference diameter in inches, and
 1.6 is the slope factor.

In our view, it is necessary to consider the characteristics of slope as two separate issues: 1) the self-thinning trajectory based on a universal slope factor, and 2) the upper boundary of the size-density relationship as indicated by observed stand conditions. We separate these characteristics on the basis that the former may be a manifestation of density-dependent mortality, whereas the latter may represent a combination of density-dependent and density-independent stand dynamics. This is an important distinction, because with few exceptions (e.g., Pretsch and Biber 2005), most analyses of the size-density relationship are based on one-time observational data, and not on long-term repeat measurements. As a result, many SDI analyses employ space-for-time substitution.

The issue of a universal slope factor (i.e., the exponent that represents the self-thinning trajectory is always 1.6) will largely be avoided in this discussion for three reasons: 1) it can be argued that a slope factor of 1.6 is inherent in Reineke's (1933) definition of SDI, and therefore immutable; 2) like the maximum SDI example above, it can be shown that some differences in slope estimation are possible solely due to the characteristics of the data used in the analysis (e.g., finding different slopes for two populations of the same forest type, one of which consists of relatively young stands and the other of which includes a broad range of age classes); and 3) although the self-thinning trajectory is generally assumed to be in effect throughout stand development, broad-scale data suggest that this assumption may be incorrect (Shaw and Long 2007). For the sake of further discussion, we will assume that the "self-thinning law" is indeed a law, but we also consider the possibility that it may not be in operation at all stages of stand development or under all stand conditions (e.g., in stands at low relative densities or experiencing non-density-related mortality).

Reasons #2 and #3, then, are related to the observation that most stands fail to maintain the expected maximum density in a mature state. This situation may be more common than is currently appreciated. FIA data collected over a wide range of stand conditions have shown that relative density begins to fall off in many forest types after quadratic mean diameter reaches about 10 inches (fig. 3, left panels). Reineke (1933) suggested that most species conformed to the "universal"

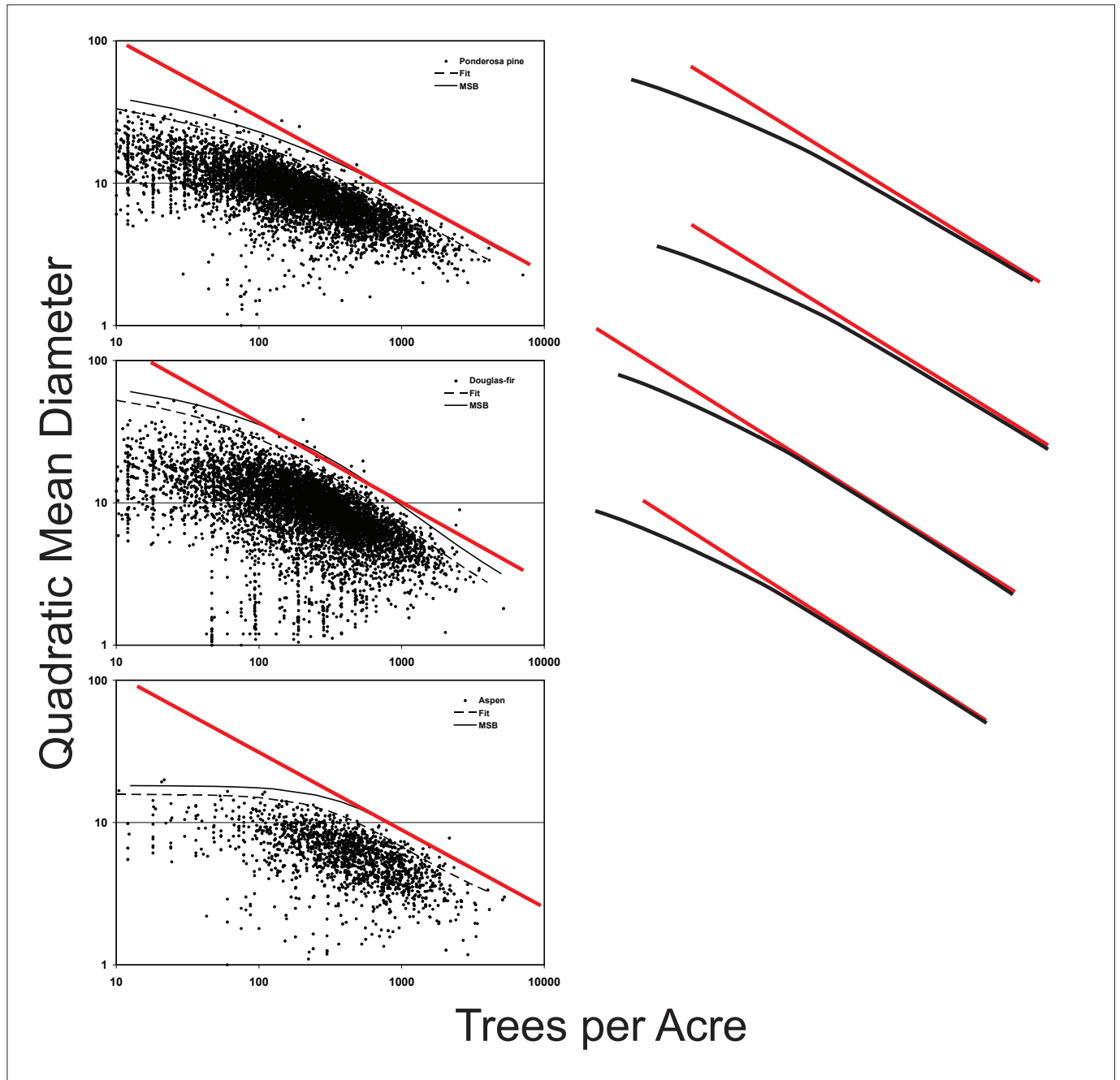


Figure 3—Size-density data showing relative density fall-off for several forest types. Left panels are Forest Inventory and Analysis data for ponderosa pine, Douglas-fir, and aspen. Solid black lines represent the approximate limit of size-density observed in natural stands, described by Shaw and Long (2007) as the mature stand boundary. Red lines represent the expected self-thinning line as defined by SDI. The right panel is taken from figure 7 in Reineke (1933), where the image has been rotated and mirrored to match the orientation of the left-side graphs. From top to bottom the graphs show data from shortleaf pine, slash pine, longleaf pine, loblolly pine from temporary plot, and loblolly pine from remeasured plots. Reineke only acknowledged divergence from the expected slope in the shortleaf pine and slash pine data (thin black lines); longleaf pine and loblolly pine data were considered to conform to the common self-thinning slope. However, in this orientation it appears that relative density falls off when quadratic mean diameter is greater than 10 inches (thick black lines) for loblolly and longleaf pines.

self-thinning trajectory, but acknowledged that data for a few species showed some degree of divergence (fig. 3, right panels). He interpreted this divergence by presenting an altered linear slope. However, when nonlinear curves are applied to the frontier of Reineke's data, it is apparent that the data he used revealed the same fall-off in relative density seen in FIA data. Reineke's (1933) data for longleaf pine therefore appear to be consistent with the mature stand boundary described by Shaw and Long (2007).

The evidence for fall-off of relative density suggests that the assumption of linearity over the entire range of size-density is incorrect. A more precise characterization would be to say that density-dependent mortality (self-thinning) may only be in effect during a certain period of stand development, and that there is a transition point where other factors limit relative density. One hypothesis for the cause of the mature stand boundary is that the mortality rate of large trees exceeds the capacity of the residual stand to capture the available growing space. This characterization of relative density transition is compatible with the Oliver and Larson (1990) stand development model, with maximum SDI limiting stand density during the stem exclusion stage and the mature stand boundary occurring during the understory re-initiation and old-growth stages.

Lack of Consistent Definitions

Another question that might be asked by the silviculturist is: What SDI should be applied in the case of irregularly structured or mixed-species stands? Because the title of Reineke's (1933) paper specified "even-aged forests," the purist approach suggests that the index is inappropriate for use in irregularly structured stands. However, Stage (1968) illustrated how SDI could be partitioned, and this approach was further developed for irregularly structured stands by Long and Daniel (1990). More recent literature on the subject (Ducey 2009; Ducey and Larson 2003; Shaw 2000) has dealt with the implications of calculating SDI by summation (equation 3), so there appears to be adequate information available to guide practitioners on how to use SDI with irregular stand structures. The alternative formulation of SDI in equation [3] therefore provides a more flexible application of the index, and avoids some issues related to the comparison of relative density in even-aged and irregularly structured stands (Shaw 2006).

$$SDI_{sum} = \sum \left(TPA_i \cdot \left(\frac{D_i}{10} \right)^{1.6} \right) \quad [3]$$

where D_i is the breast height diameter of the i th tally tree on the plot and
 TPA_i is the number of trees per acre represented by the i th tree.
 10 is the reference diameter in inches, and
 1.6 is the slope factor.

Less attention has been paid to the question of species mixtures, although various approaches to weighting maximum SDI by stand composition have been discussed by Puettmann and others (1992), Dean and Baldwin (1996), Torres-Rojo and Martinez (2000), and Woodall and others (2005). Although the approaches vary, these studies and examination of FIA data indicate that composition is a factor in the determination of potential maximum SDI for a stand. All of these studies suggest that the appropriate "adjustment" for SDI in mixed-species stands should be based on weighting of the individual species maximum SDI, as determined through analysis of pure stands, by the relative composition of each species in the mixed stand. In other words, the expected maximum SDI for a 50:50 mixture of two species should be approximately the average of the individual species maxima. If such a stand is a candidate for thinning and one species will become

more dominant than the other after treatment, then a new maximum should be calculated for the resulting stand based on the post-thinning proportions.

The questions about effects of structure and mixtures are relevant to the issue of consistency in the development and use of SDI. Most studies on SDI are silent with respect to the ranges of composition and structure that are represented in the analyzed data. Because both stand characteristics have an effect on SDI calculations, they should be considered in research used to estimate maximum SDI or develop tools such as density management diagrams, and they should be considered by the silviculturist when designing density management regimes. As with the plot scale issue, problems may be caused for SDI users simply because of a lack of stated or consistent definitions as part of research methods. For example, if two studies are designed to determine the maximum SDI for a species, but one uses data largely acquired from pure stands (e.g., >90% composition of the target species) and the other uses data acquired from stands where the species is merely dominant (e.g., >50% composition of the target species), there is a high likelihood that they will find two different SDI maxima for the same species. Without explicit definition of the analyzed population, users may be confused by these results.

Proposals for Future Research and Application

In the discussion above we describe four factors that can cause inconsistencies between values of SDI as determined in research studies and field application: 1) sampling error, 2) scale at which the data are obtained (plot scale), 3) assumption of linearity of the self-thinning dynamic during advanced stages of stand development, and 4) lack of consistent definitions or a mismatch between the population used to develop SDI-related values and the population being assessed in management practice. We will not discuss sampling error here, because it is an inevitable artifact of estimation and a minor issue in comparison to the others. We believe that the remainder of the issues can be resolved through the adoption of a few basic guidelines for use during research and field application of SDI.

Researchers should ensure the data used in SDI-related analyses are compatible with the scale expected to be used by the practitioner when the results of research are applied in the field. Methods or recommendations for application should state the appropriate scale of application. This is not to say that data of different scale are not useful in the research process. Small-scale (plot-level) data may be useful to determine the biological capacity of a species for packing on the site, whereas larger scale (stand-level) data are more likely to describe the range and variability of density that would be measured at management scales. Both scales of information might be useful to the silviculturist, who might simultaneously seek to manage for dense pockets and more moderate stand-level density. In addition to matching of source data and application scales, it is important to define the population to which the results apply. The population of interest is commonly thought of in terms of geographic extent, but the description should include both the range of composition and stand structure. Scale, composition, and structure have the potential to affect determination of maximum SDI, and it is likely that one or more of these characteristics have been a factor in cases where there are apparent regional differences among maximum SDI values. If the population of interest is adequately identified, then apparent regional differences should be minimal.

The upper boundary of the size-density relationship is complex, so there is little to be gained through continued evaluation of the self-thinning slope where a linear relationship is assumed. Instead, research should focus on the mature stand boundary, including mechanisms underlying the fall-off phenomenon and stand

dynamics when stem exclusion is not the driving factor. Increasing knowledge in this area should substantially improve the realism of stand projection models such as the Forest Vegetation Simulator.

Finally, a definition of maximum SDI should be consistent with “benchmark” percentages of the maximum that correspond to canopy closure, the onset of competition, and the zone of imminent competition mortality (Drew and Flewelling 1977). This compatibility is required in order for density management regimes to produce desired results. If density is scaled incorrectly in tools and growth models, then silviculturists risk missing management objectives because density may inadvertently be managed higher or lower than the density target would indicate. This is the case that led Basford (this proceedings) to abandon SDI in favor of an alternative. Anyone attempting to manage ponderosa pine using a maximum SDI of 830, when 450 is the appropriate maximum, is likely to fail to meet their objectives because the true relative density (and by extension, the state of competition) in the managed stands would always be much higher than suggested by the SDI percentage.

Conclusion

In this paper we have attempted to highlight inconsistencies in research and application of SDI, and suggested opportunities for resolution of the issues causing the most confusion when applying the index in silvicultural practice. Most of these issues may be resolved simply by full disclosure of the data and methods used for analysis, coupled with use by the practitioner that is consistent with the results. In some cases where inconsistencies have been found, they may be traced back to weakly supported conclusions. For some reason, an incorrect maximum SDI for one of the most important timber species in the West, ponderosa pine, went unquestioned for over 75 years. This might not be surprising, considering the foundational nature of Reineke's work, but the persistence of this unsupported value in Forest Service documentation suggests that it was never examined closely. We believe that situations like this can be avoided in the future, if SDI research and application are conducted with the adoption of a few simple guidelines:

- Density management guidelines should be developed using spatial scales comparable to what would be used in application.
- Research documentation should explicitly describe the population from which the data were drawn, in terms of composition and structure, so that practitioners understand the range of conditions represented in the analysis.
- Definitions of terms used in research should be consistent with accepted definitions, or explicitly defined in cases where they are open to interpretation.

Adoption of standards and guidelines such as these, especially in cases where the same concept may be described in many different geographic areas and forest types using multiple methods, should do much to avoid confusion during technology transfer from researcher to the practitioner.

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