Estimating maximum allowable timber yield by Linear Programming

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U. S. FOREST SERVICE RESEARCH PAPER NE-17
1964
NORTHEASTERN FOREST EXPERIMENT STATION, UPPER DARBY, PA.
FOREST SERVICE, U. S. DEPARTMENT OF AGRICULTURE
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About the author . . .

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Introduction

FORESTERS, like many other professional or business men, are very often interested in extreme rather than average situations. For example, they strive for maximum dollar returns, growth, and quality; or, conversely, they strive for minimum costs, mortality, and defect. Sometimes the course of action necessary to achieve a maximum or minimum situation is quite evident. However, under other circumstances the possible courses of action may be so numerous or conflicting that the best choice among them cannot be reasoned out. In such complicated situations, linear programming may serve as a useful analytical tool.

Linear programming is a mathematical technique that provides a maximum or minimum solution to a linear equation when the variables in the equation are restricted within certain limits. Many common situations can be expressed in the form of simple linear equations. For example, the weekly grocery bill can readily be expressed as a linear sum of the quantity of each food item times its unit cost; and here the quantities are the variables and the unit costs are the constants in the equation. Carrying this example one step further, the concept of limitations upon the variables is easily comprehended: one might restrict the amount of each type of food to a certain weekly quota, or might restrict the amount spent to certain dollar values so as to achieve a reasonably balanced diet.

In forestry, linear programming so far has been used primarily for analyzing economic problems of allocation. The technique has seldom, if ever, been applied to the solution of problems in mensuration, silviculture, or protection, possibly because few foresters are familiar with its use and limitations. However, several common forestry problems appear to be amenable to linear programming. An approach to the solution of one of these problems, the estimation of maximum allowable timber yields, will be described in this paper.

General Equations

Assuming an unbalanced even-aged forest comprised of unequal proportions by area of several age classes, the yield of that forest (from clearcutting, for example, coupled with intermediate thinnings,) over the first operating cycle[1] equals the sum of the area cut in each age class times the harvestable volume per acre plus the area thinned in each age class times the volume thinned per acre. Similarly, the total yield of the forest over a rotation equals the sum of the yields in all operating cycles within the rotation. Thus, total yield over a rotation can be expressed as the following linear equation:

$$[1] \qquad Y = \Sigma\Sigma A_{ij} V_{ij} + \Sigma\Sigma B_{ij} T_{ij}$$

in which—

Y = Total yield.
A = Area harvested (acres).
V = Volume per acre harvested.
B = Area thinned (acres).
T = Volume per acre thinned.
i = The number of the operating cycle (first, second, third, etc.)
j = The age class.

[1] In this paper *operating cycle* is defined as the number of years required to examine all portions of the forest and make harvest cuttings or thinnings as necessary.

Now there are many different ways in which an unbalanced even-aged forest can be cut. Some foresters might prefer to make harvest cuttings in only one or two of the older age classes, and to make thinnings in the remaining age classes. Other foresters might prefer to do some harvest cutting in the younger age classes so as to work toward a better age-class distribution. But despite divergent opinions, most foresters would agree that one main objective of regulation is to maximize the yield of the forest—whether measured in dollars or in volumes. Thus a reasonable objective for a linear programming analysis is to maximize yields as expressed by the above yield equation. In linear programming terminology, this yield equation is called the objective equation or objective function.

But maximum yields are not the only objective of regulation. We might also wish to make sure that approximately equal volumes are harvested during each operating cycle. To introduce this restriction into the linear programming solution, it would be necessary first to express it in an equation of the form:

$$\Sigma A_{1j} V_{1j} = \Sigma A_{2j} V_{2j} \qquad [2]$$

in which the variables and subscripts are as previously defined. This equation states that the volume cut during the first operating cycle will equal the volume cut during the second cycle. Similar equations can be used to equate the harvests among other pairs of operating cycles.

Another useful restriction on the problem might be that equal areas are harvested in each operating cycle so as to bring about a balanced age distribution by the end of the first rotation. This restriction would be expressed by an equation of the form:

$$\Sigma A_{1j} = \Sigma A_{2j} \qquad [3]$$

which states that the area harvested during the first operating cycle equals the area harvested during the second cycle. For this restriction to be effective, it would also be necessary to specify that no subsequent harvest cutting will take place in the newly created age classes during the remainder of the rotation.

Another very necessary restriction on the problem is that areas harvested and thinned in any age class will be equal to or less than the available area. For example, if we began with 100 acres in the 40-year age class, it would be necessary to specify that the area harvested in this age class during the first operating cycle will be no more than 100 acres, and that the area harvested in the 50-year age class during the second cycle (assuming a 10-year cycle) will be no more than 100 minus the area cut during the first cycle. These two relationships can be expressed, respectively, by the equations:

$$[4] \qquad A_{14} \leq 100$$

$$[5] \qquad A_{25} \leq 100 - A_{14}$$

In actual practice, the problem will probably be much more complicated than the general one considered above. An unbalanced even-aged forest might be divided into sites, types, and stocking classes as well as age classes. In such cases, the objective equation and the restrictions would be considerably more involved. Furthermore, many additional restrictions could be written into the problem, depending on the particular management situation and objectives. However, the equations and examples presented so far provide a fairly representative picture of how the problem of yield estimation can be approached through linear programming.

Solving the equations that comprise a linear programming problem is an iterative and often lengthy process that requires some facility with simultaneous equations. A simple description of linear programming theory may be found in Kemeny, Snell, and Thompson,[2] and a more advanced discussion of the theory and computations in Boulding and Spivey.[3] The details will not be covered here. For many practical problems, an electronic computer is necessary; and fortunately the necessary computer programs are available.

Three types of solution are possible: (1) an optimum solution giving the values of the variables that will maximize (or minimize) the objective equation; (2) an infinite solution, which occurs when the restrictions on the problem do not set an upper

[2] Kemeny, John G., J. Laurie Snell, and Gerald L. Thompson. INTRODUCTION TO FINITE MATHEMATICS. 372 pp., illus. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1956.

[3] Boulding, Kenneth E., and W. Allen Spivey. LINEAR PROGRAMMING AND THE THEORY OF THE FIRM. 227 pp., illus. Macmillan Co., New York, 1960.

(or lower) limit to the value of the objective equation; and (3) no solution, which results when the restrictions on the problem are not consistent with one another. It is very likely that examples of no solution—as well as optimum solution—will occur when linear programming is used in the estimation of maximum allowable yields. In such cases, it will be necessary to examine the restrictions for consistency, to revise them as necessary, and then to try again.

Two Hypothetical Examples

Linear programming has not yet been applied to a practical problem of yield estimation. No doubt difficulties will be encountered that are not yet evident. However, the solution of two simple hypothetical examples can be given to further illustrate the possibilities of linear programming.

Assume that an unbalanced even-aged forest has three 10-year age classes represented by unequal areas:

Age class, years		
0-10 (acres)	*10-20 (acres)*	*20-30 (acres)*
60	30	10

And assume further that the rotation age is 30 years, and that we wish to maximize cubic-foot yields over three 10-year operating cycles subject to the restrictions of: (1) equal areas cut per operating cycle; (2) a balanced age-class distribution by the end of the rotation (equal areas in each age class); and (3) areas cut by age classes not to exceed the available acreages. To keep the problem simple, we will eliminate thinning yields from consideration. Hypothetical harvest yields per acre, as might be estimated from a standard or empirical yield table, are given in table 1. In a practical problem, the construction of this yield table so as to reflect present and expected net harvestable volumes per acre would be a very critical step in the yield analysis.

The objective equation for this problem is derived from equation (1); thinning yields are omitted, and hypothetical harvest yields from table 1 are inserted as constants. We begin with the equation:

$$\begin{aligned}\text{Yield} = {} & A_{11}\,(800) + A_{12}\,(1600) + A_{13}\,(3000) \\ & + A_{21}\,(1100) + A_{22}\,(2000) + A_{23}\,(3200) \\ & + A_{31}\,(1200) + A_{32}\,(2200) + A_{33}\,(3600)\end{aligned}$$

To ensure that the forest will develop toward a balanced age distribution, it is necessary to eliminate all cutting in new stands established during the three operating cycles. This is accomplished

Table 1.—*Hypothetical harvest yields, in cubic feet per acre, by age class and operating cycle*[1]

Operating cycle	Age class, years		
	0-10	10-20	20-30
1	800	1,600	3,000
2	1,100	2,000	3,200
3	1,200	2,200	3,600

[1] These cubic-foot yields were chosen for convenience and are not intended to be realistic. The increase in yields by age class from one operating cycle to another is meant to represent improvement of the stand in net volume resulting from cultural work.

Table 2.—*Area available for cutting and area cut, by age class and operating cycle*

(Primary restriction: development of a balanced age distribution)

Operating cycle	Age class, years						Total cut	
	0-10		10-20		20-30			
	Available	Cut	Available	Cut	Available	Cut	Area	Volume
	Acres	*Acres*	*Acres*	*Acres*	*Acres*	*Acres*	*Acres*	*Cu. ft.*
1	60.00	23.33	30.00	0.00	10.00	10.00	33.33	48,664
2	33.33	.00	36.67	3.33	30.00	30.00	33.33	102,660
3	33.33	.00	33.33	.00	33.34	33.34	33.34	120,024
Total	—	—	—	—	—	—	100.00	271,348

by eliminating appropriate terms from the above equation; namely, the terms involving A_{21}, A_{31}, and A_{32}. Furthermore, if we impose the condition that all of the area in the third age class will be harvested during each operating cycle, three more variables can be eliminated from the objective equation because they can be expressed as:

$$A_{13} = 10$$
$$A_{23} = 30 - A_{12}$$
$$A_{33} = 60 - A_{11} - A_{22}$$

By eliminating terms and variables as explained above and simplifying, the final objective equation is:

$$\text{Yield} = A_{11}\ (-2800) + A_{12}\ (-1600) + A_{22}\ (-1600) + 30{,}000 + 96{,}000 + 216{,}000$$

The restriction of equal areas cut per cycle is expressed by two equations in the form of equation 3. The areal restrictions on the three variables in the objective equation follow the form of equations 4 and 5.

The linear programming solution yielded the area values given in table 2. These values represent the areas to be cut by age class and operating cycle so as to provide maximum yields over a rotation under the specified restrictions of the problem. Equal areas are to be cut during each operating cycle and a balanced age-class distribution will be attained by the end of the rotation as specified in the restrictions of the problem. However, the volume yields will vary considerably from one operating cycle to another.

We may rephrase the restrictions of the problem so as to ensure equal volume yields per operating cycle with no limitations upon areas cut except those imposed by the initial available acreages in each age class.

Once again, the objective equation is derived from equation 1. This time, no complete terms are omitted, but three variables can be eliminated as before and some simplification can be done to provide the following yield equation:

$$Y = A_{11}\ (-2800) + A_{12}\ (-1600) + A_{21}\ (1100) + A_{22}\ (-1600) + A_{31}\ (1200) + A_{32}\ (2200) + 30{,}000 + 96{,}000 + 216{,}000$$

Table 3.—*Area available for cutting and area cut, by age class and operating cycle*

(Primary restriction: equal volumes cut per operating cycle)

Operating cycle	Age class, years						Total cut	
	0-10		10-20		20-30			
	Available	Cut	Available	Cut	Available	Cut	Area	Volume
	Acres	*Acres*	*Acres*	*Acres*	*Acres*	*Acres*	*Acres*	*Cu. ft.*
1	60.00	60.00	30.00	25.67	10.00	10.00	95.67	119,072
2	95.67	95.61	.00	.00	4.33	4.33	99.94	119,027
3	99.94	99.12	.06	.06	.00	.00	99.18	119,076
Total	—	—	—	—	—	—	294.79	357,175

The restriction of equal volume yields per operating cycle is expressed by two equations in the form of equation 2 while the areal restrictions are again represented by equations such as 4 and 5.

The solution to this problem is given in table 3. By cutting the indicated areas, maximum yields can be obtained within the restrictions of the problem. Not only are the volume yields to be equal—except for rounding errors—for all operating cycles, but total volume yields will be much greater than in the first problem. However, a balanced age-class distribution will be sacrificed for higher and more uniform yields. Clearly, it will be desirable to set or attempt to set restrictions upon both volume yields and age-class distribution when applying linear programming to a practical problem of regulation.