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Structural stocking guides: a new look at an old friend

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Abstract: A parameter recovery-based model is developed that allows the incorporation of diameter distribution information directly into stocking guides. The method is completely general in applicability across different guides and forest types and could be adapted to other systems such as density management diagrams. It relies on a simple measure of diameter distribution shape, the basal area larger than quadratic mean stand diameter, to estimate the parameters of the unknown distribution. This latter quantity is shown to have high correlation with stocking guide variables in northeastern forest types. A primary objective of this new type of guide is that its use should require a minimal amount of new information from the user and that the underlying model should be as simple as possible.

Résumé : Un modèle basé sur la récupération de paramètres a été développé pour permettre l'incorporation d'information sur la distribution des diamètres directement dans des guides de densité relative. La méthode est totalement générale et applicable pour les différents guides et types forestiers ; elle pourrait être adaptée à d'autres systèmes comme les diagrammes de gestion de la densité. Elle repose sur une simple mesure de la forme de la distribution des diamètres, la surface terrière cumulée des arbres des catégories de diamètre supérieures au diamètre moyen quadratique du peuplement, pour estimer les paramètres de la distribution inconnue. Cette dernière variable est fortement corrélée aux variables des guides de densité dans les types forestiers du Nord-Est. Le principal objectif de ce nouveau type de guide consistait à réduire au minimum l'information nouvelle requise par son utilisateur et à garder le modèle sous-jacent aussi simple que possible.

[Traduit par la Rédaction]

Introduction

Stocking guides were first introduced by Gingrich (1967) as a means for aiding the decision-making process in silvicultural stand prescription activities. Since the introduction of Gingrich's first upland oak stocking guide, numerous other guides have been constructed for many forest types in the United States and Canada. While there are some differences in the methods of preparation for subsequent guides, most stocking guides follow the general format of Gingrich (1967) for presentation (Leak 1981). The guide itself is set in the cartesian plane described by the number of trees per acre (N) (1 acre = 0.404 685 ha) on the x axis and basal area per acre (B) on the y axis; hereafter this is referred to as the N–B plane. Additional lines denoting stands of constant quadratic mean stand diameter ($\bar{D}_q$) connecting the lines of average maximum stocking (A-line) and minimum stocking for full site utilization (B-line) are also included. Sometimes, but not always, a C-line is added, which represents the collective set of stand conditions in the N–B plane that will grow to the B-line in 10 years under average site conditions.

There has been much discussion over how best to fit the different stocking lines on the guide and what kinds of stands should be used in so doing. In addition, some researchers have raised pertinent questions about what stocking guides truly represent in relation to stand growth (e.g., Leak 1981). Others have advocated adding related auxiliary information to stocking guides. For example, Seymour and Smith (1987) added lines of constant height to the original eastern white pine (*Pinus strobus* L.) stocking guide. The model developed for their guide also enabled them to suggest a new B-line formulation, which was well below that of the original. Growth information has also been added to stocking guides by Leary and Standfield (1986) in the form of direction fields. It was proposed that these dynamic stocking guides allowed one to project the future position of candidate stands on the N–B plane by following the direction field for that stand as shown on the guide. Initially, these fields were set up for 10-year projections, but any time interval could be used. Goelz (1990) developed a similar strategy for incorporating growth onto stocking guides; however, rather than using direction fields, contours were plotted on the N–B plane representing 10-year volume growth.

In recent years, stand density management diagrams (DMDs) (Drew and Flewelling 1977, 1979) have gained in popularity both in North America and worldwide, finding their roots in the Japanese literature based on the principle of self-thinning (Yoda et al. 1963). These guides may be constructed with mean tree volume, biomass, or $\bar{D}_q$ (e.g., McCarter and Long 1986) on the y axis rather than basal area per acre, while N remains on the x axis. If judged by the sheer number of extant DMDs, the guides based on mean tree volume are evidently preferred by foresters. However, it is easily shown that those based on $\bar{D}_q$ are closely related to stocking guides and also Reineke's (Reineke 1933) stand density index. Like stocking guides, DMDs often portray other information, such as lines of constant $\bar{D}_q$ (on the mean tree volume guides) and height (Drew and Flewelling 1979;

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Newton and Weetman 1993).

The intent of this paper is not to enter the debate over whether stocking guides or DMDs should be preferred, or even how or what information should be presented on such guides. Rather, recognizing the fact that stocking is also intrinsically tied to stand structure in the form of the stand diameter distribution, a modeling technique is proposed that can be used to add information on the underlying distribution to such guides. Stocking guides are chosen for illustration, but similar models can be developed for DMDs as well. The proposed modeling strategy uses parameter recovery (PR) (Hyink and Moser 1983) methods to estimate the parameters of the underlying distribution at any point on the N–B plane. The joint prime objective was to develop the simplest model possible for illustration of the technique tailored to an individual stocking guide, while requiring a minimal amount of new information from the user of such guides.

There are numerous situations in which information about the diameter distribution for a stand may be lacking, while having knowledge of such information would be useful. For example, following the direction field trajectories on dynamic stocking guides places a stand at a new projected point 10-years hence with no associated distributional information — while an estimate of the diameter distribution for the current stand may be known from a prescription inventory, the future distribution remains unknown. The same would be true for any whole-stand growth model that might be used for projecting the growth of existing stands on the guide. Alternatively, a stand may be in need of a treatment in the form of a thinning at the present time. For the sake of example, it might be expedient to move the stand from the current position on the N–B plane to the B-line, while maintaining the same $\bar{D}_q$ for the stand. The diameter distribution for the target stand is unknown at this point. Knowledge of even an estimated diameter distribution for the target stand could simplify marking in the case of upcoming silvicultural activities. It may even be useful from a pedantic point of view to compare the existing stand conditions to some smooth theoretical model that corresponds to that stand. Finally, it may simply be of interest to visualize how stand structure varies over a given stocking guide. In all cases, the proposed model for adding diameter distribution information to stocking guides would provide a solution.

Parameter recovery methods

Parameter recovery models were first introduced by Hyink (1980) and later formalized by Hyink and Moser (1983). Parameter recovery models are normally built into or coupled with systems of equations for growth and yield modeling on a whole-stand basis (e.g., Lynch and Moser 1986; Murphy and Farrar 1988). In this framework, stand-based quantities that can be mathematically predicted by the growth models are set equal to corresponding quantities as given by a probability density function (PDF); one equation is established for each unknown parameter in the PDF being used. This yields a system of typically nonlinear equations that can be solved simultaneously for the unknown PDF parameters — in concept, PR methods are very similar to the method of moments used in classical statistics. However, parameter recovery methods need not be explicitly coupled to growth projection systems; observed quantities can be used as well. In the parameter recovery model presented here, a mixture of known (possibly with sampling error) and predicted quantities will be employed.

The entirety of information required for the use of classical stocking guides can be determined by the familiar triplet of basal area per acre, number of trees per acre and the quadratic mean stand diameter. For convenience, this triplet may be denoted as $\boldsymbol{\psi} = (B, N, \bar{D}_q)$. Stocking guides allow the graphical representation of these variables in two dimensions because of their intrinsic relation

$$[1] \qquad \bar{D}_q = \sqrt{\frac{B}{N\kappa}}$$

where κ is the conversion factor from inches (1 in. = 2.54 cm) to square feet (1 ft^2 = 0.093 m^2). This relationship may be used to advantage to determine a parameter recovery model for structural stocking guides — at least in part.

The two-parameter Weibull distribution was chosen for this parameter recovery model because of its long history of use in forestry. More importantly, however, the two-parameter Weibull retains a great amount of inherent flexibility of form for a distribution parameterized by only two unknowns. Because parameter recovery models require one estimating equation for each unknown parameter, this becomes a very important consideration, given the initial objective of requiring virtually no new information from the forest manager. Given the constraints on the information content in eq. 1, the number of unknown parameters becomes a very important consideration in the model. Because stocking guides are presented in the cartesian plane and require only two of the three stand variables found in $\boldsymbol{\psi}$ to locate any stand, it is tempting to think that two independent equations can be generated for parameter recovery from this relationship. However, this is not the case. A little further reflection reveals that once one of the stand variables is determined, both of the two remaining stand variables are invariant for a given point on the stocking guide. Thus, the use of eq. 1 allows only one degree of freedom in the determination of estimating equations for the parameter recovery scheme. As a consequence, we must look elsewhere for the second equation.

In this study, the random variable D is diameter at breast height (DBH). The DBH-frequency distribution can be characterized by the two-parameter Weibull, which is given as

$$f(d; \boldsymbol{\theta}) = \left(\frac{\gamma}{\beta}\right)\left(\frac{d}{\beta}\right)^{\gamma-1} \exp[-(d/\beta)^{\gamma}], \qquad d, \gamma, \beta > 0.$$

The two unknown parameters in this distribution are the shape parameter γ and the scale parameter β and comprise the parameter vector $\boldsymbol{\theta}$. In addition, the αth raw moment of the two-parameter Weibull is $\mu'_\alpha = \beta^\alpha \Gamma(\alpha/\gamma + 1)$. This last relationship becomes critical to the development of the first parameter recovery equation.

The basal area equation

A special case of the raw moment equation for the two-parameter Weibull occurs when $\alpha = 2$; viz

$$\mu'_2 = \int d^2 f(d; \boldsymbol{\theta})\, \mathrm{d}d$$

and, as has been noted before (Ek et al. 1975; Burk and Newberry 1984; Gove and Patil 1998), $\mu_2' = \bar{D}_q^2$. In addition, basal area per acre can be calculated using the moment method (Gove and Patil 1998) as

$$B^\mu = N\kappa \int d^2 f(d;\boldsymbol{\theta})\,\mathrm{d}d = N\kappa \bar{D}_q^2$$

Assembling these facts and letting $k_2 = 2/\gamma + 1$ leads to the following equation:

$$B^\mu = N\kappa\beta^2\Gamma(k_2)$$

where, in general, $\Gamma(k) = \int_0^\infty t^{k-1}\exp(-t)\,\mathrm{d}t,\ k > 0$ is the gamma function. This allows development of the first parameter recovery equation. First, however, let $\boldsymbol{\psi}_s = (B_s, N_s, \bar{D}_{q_s})$ describe the necessary stand parameters for any point s on the stocking guide for which a diameter distribution is desired. Therefore, the basal area, number of trees, and quadratic mean stand diameter are assumed known (for the sake of model development only, we assume these quantities are known without error at this point) ahead of time for the stand. Setting the basal area for the stand at stocking guide point s equal to the above equation yields the first parameter recovery equation; viz

[2] $$B_s = B^\mu$$

It is unfortunate that the interdependence of the stand variables in $\boldsymbol{\psi}$ is such that no matter how ingenious, any attempt to write another independent parameter recovery equation in terms of the one of the remaining variables will fail. Any attempt to solve such a system results in infinitely many solutions for any given point on the stocking guide; the solutions trace out the path of the two-parameter Weibull distribution in the Pearson (skewness-kurtosis) plane. Thus, there is no intrinsic diameter distribution associated with any given point on a stocking guide given the currently available information in $\boldsymbol{\psi}$. The requirement then becomes one of finding a measure that will essentially fix the shape of the Weibull at any point s to make up the second parameter recovery equation.

The BALM equation (the basal area larger than the quadratic mean stand diameter)

An interesting alternative to the usual parameter recovery scheme is to mix in what amounts to a percentile-based estimating equation. In this case, one could consider the proportion of trees per acre in some portion of the diameter distribution, or alternatively, one could choose some proportion of basal area. The two important requirements are that (*i*) the chosen area in the distribution is useful in defining the shape of the distribution and (*ii*) one of the other variables in $\boldsymbol{\psi}$ must be present in the estimating equation in order to fix the specific point on the stocking guide. Auxiliary considerations include the fact that the quantity chosen must be easily understood by foresters and simple to calculate. A quantity that has evidently not received much previous attention is useful for this purpose: BALM.

It turns out that BALM has a number of useful characteristics, not the least of which is its ease of calculation. Given an empirical diameter distribution, say from the stand table in a forest inventory, BALM can be easily calculated as

$$\mathrm{BALM}_{\mathcal{E}} = \kappa \sum_{d \ge \bar{D}_q} d^2.$$

In addition, BALM can also be calculated from either the DBH-frequency distribution or basal area-size distribution (Gove and Patil 1998) as

$$\mathrm{BALM}_{\mathcal{F}} = N\kappa \int_{d \ge \bar{D}_q} d^2 f(d;\boldsymbol{\theta})\,\mathrm{d}d$$

$$\mathrm{BALM}_{\mathcal{B}} = B \int_{d \ge \bar{D}_q} f_2^*(d;\boldsymbol{\theta})\,\mathrm{d}d$$

respectively, where $f_2^*(d;\boldsymbol{\theta})$ is the size-biased PDF of order $\alpha = 2$ for the basal area-size distribution. Following Gove and Patil (1998), under the two-parameter Weibull model these two equations can be written in terms of the gamma function, $\Gamma(k)$, and the incomplete gamma function, $(\gamma(k,x) = \int_0^x t^{k-1}\exp(-t)\,\mathrm{d}t,\ k > 0)$, as

$$\mathrm{BALM}_{\mathcal{F}} = N\kappa\beta^2\xi$$

$$\mathrm{BALM}_{\mathcal{B}} = \frac{B\xi}{\Gamma(k_2)}$$

where $\xi = \left[\Gamma(k_2) - \gamma(k_2, (\bar{D}_q/\beta)^\gamma)\right]$. This last set of equations are in a form convenient for computation.

To put BALM on a more useful basis for parameter recovery, recall that the discussion was originally motivated by specifying a proportion of the overall basal area per acre that can be attributed to BALM. Thus, proportion BALM, defined as $\propto\mathrm{BALM} = \mathrm{BALM}/B$, is a more appropriate quantity. Graphically, the relationship among $\propto$BALM, $\bar{D}_q$, and the diameter distributions $f(d;\boldsymbol{\theta})$ and $f_2^*(d;\boldsymbol{\theta})$ is illustrated in Figure 1. Notice that $\propto$BALM is simply the area under the curve for all diameters larger than $\bar{D}_q$ in the basal area-size distribution. In addition, it can be shown that the two curves always intersect at the quadratic mean stand diameter (Gove 2003*c*).

Formulas for $\propto$BALM follow directly from the relationships above. For the empirical distribution

$$\propto\mathrm{BALM}_{\mathcal{E}} = \frac{\kappa}{B} \sum_{d \ge \bar{D}_q} d^2$$

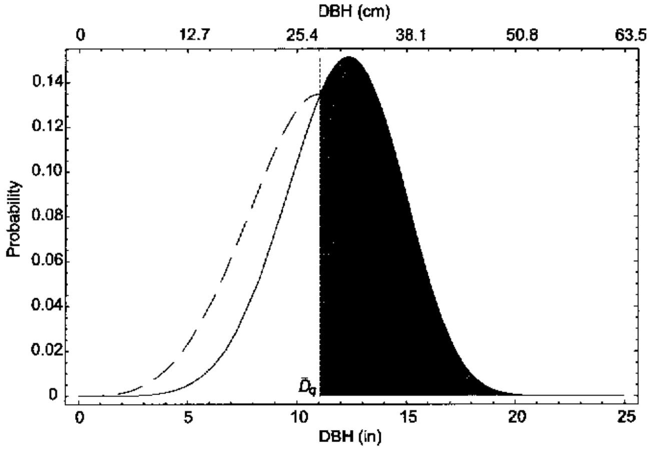
and regardless of whether the DBH-frequency or the basal area-size distribution is used, it can be shown that both will yield the same equation; viz

$$\propto\mathrm{BALM}_{\mathcal{B}} = 1 - \frac{\gamma(k_2, (\bar{D}_q/\beta)^\gamma)}{\Gamma(k_2)}$$

Notice that this latter equation also fulfills the requirement that either $\bar{D}_q$ or N be in the PDF-based portion of the parameter recovery estimating equation in order for a given point s on the stocking guide to be uniquely defined.

The final form of the parameter recovery equation for $\propto$BALM evidently requires an additional piece of information;

Fig. 1. Graphical definition of $\propto$BALM (shaded region) using the DBH-frequency (dashed) and basal area-size (solid) distributions.



that is, some estimate of $\propto$BALM for the sth point on the stocking guide in which interest lies. Let $\hat{P}_s$ represent this missing piece of information. For the present, we will assume either that $\hat{P}_s = \propto\text{BALM}_{\mathcal{E}}$ or that we have some guess or estimate of $\hat{P}_s$ available. Then the second parameter recovery equation for the two-parameter Weibull distribution is

[3] $$\hat{P}_s = \propto\text{BALM}_{\mathcal{B}}$$

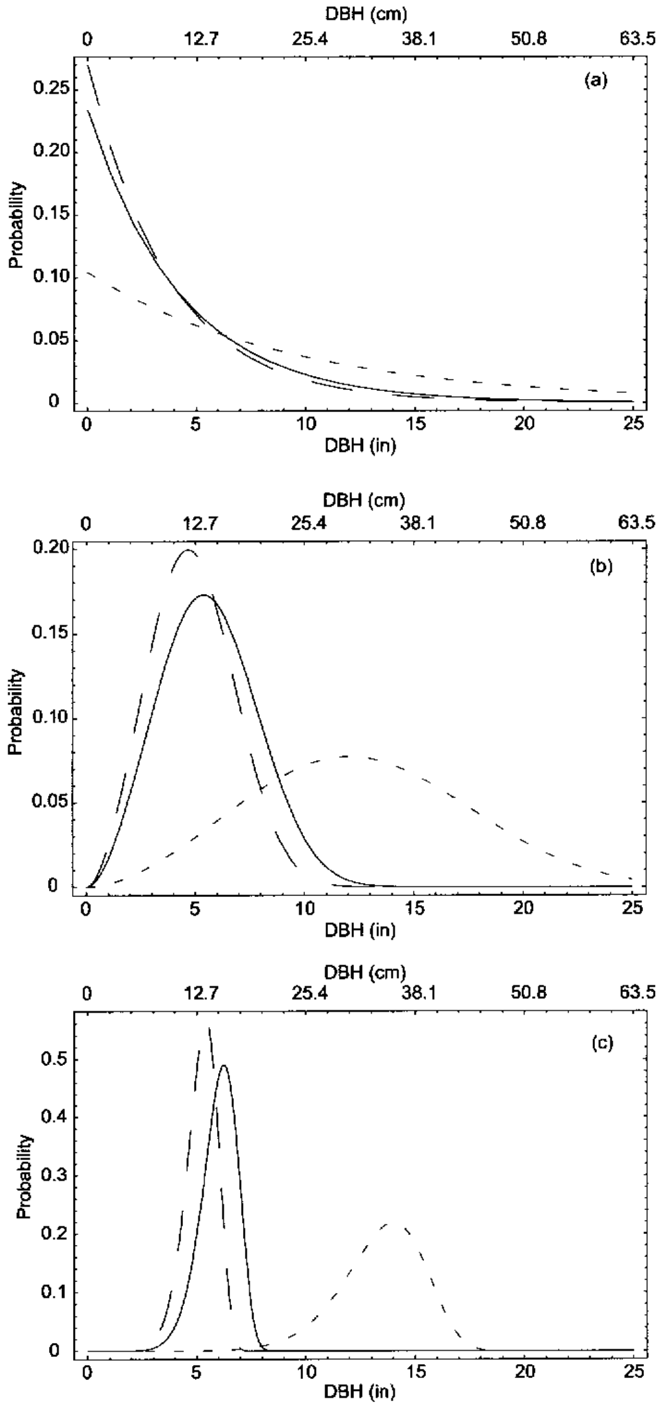
Equations 2 and 3 completely specify the parameter recovery estimating equations for structural stocking guides. These equations may be easily solved for a point s on the stocking guide. The solution can be found solving the following simple, nonlinear, minimization problem in terms of the unknown parameter vector $\boldsymbol{\theta}$ for the two-parameter Weibull distribution

[4] $$\begin{aligned} &\underset{\{\gamma,\beta\}}{\text{Min}} \quad R_1^2 + R_2^2 \\ &\text{St}: \quad B_s - B^{\mu} = R_1 \\ &\qquad\;\; \hat{P}_s - \propto\text{BALM}_{\mathcal{B}} = R_2 \end{aligned}$$

which yields an estimated parameter recovery vector $\tilde{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$.

Figure 2 illustrates that $\propto$BALM effectively determines the shape of the unknown diameter distribution in question for any given stocking guide point. In this figure, the parameter recovery model [4] was solved at several points on a stocking guide as shown in Table 1 with $\tilde{\gamma}$ and $\tilde{\beta}$ being the estimated parameters. Figures 2a–2c illustrate the situations where $\hat{P}_s = 0.83, 0.70$, and 0.62, respectively. Notice that for a given level of $\hat{P}_s$, the scale parameter for the Weibull changes according to the points on the stocking guide, but that the shape parameter remains fixed (Table 1 and Figure 2). At a constant level of $\hat{P}_s$, this remains true for any point on the guide. Thus, if $\hat{P}_s$ is specified, fixing the shape of the diameter distribution, the scale parameter varies over the entire N–B plane, determining the difference in stand structure for a given $\propto$BALM. It should be clear that there are infinitely many values of $\propto$BALM and $\boldsymbol{\psi}$ combinations covering the full range of shapes for the two-parameter Weibull PDF.

Fig. 2. An illustration of the effect of $\propto$BALM on diameter distribution shape using the data in Table 1 for stands 1 (long dashed), 2 (solid), and 3 (short dashed) at (a) $\hat{P}_s = 0.83$, (b) $\hat{P}_s = 0.7$, and (c) $\hat{P}_s = 0.62$.



Finally, because $\propto$BALM is a relative measure that relies on $\bar{D}_q$, it seems reasonable that the minimum stand diameter ($D_{\min}$) will also affect this quantity. That is, there can be many different stand conditions with different ($D_{\min}$, $\bar{D}_q$) combinations that can produce the same relative value of BALM. The importance of including $D_{\min}$ will become apparent in the next section. Combining all of the above information re-

Table 1. Parameter recovery estimates $\tilde{\boldsymbol{\theta}} = (\tilde{\gamma}, \tilde{\beta})$ for three different stands on a stocking guide corresponding to the graphs in Figure 2.

				$\tilde{\beta}$		
Stand (s)	B_s (ft^2·acre^{-1})	N_s (acre^{-1})	$\bar{D}_{q_s}$ (in.)	$\hat{P}_s = 0.83$ $\tilde{\gamma} = 1.00$	$\hat{P}_s = 0.70$ $\tilde{\gamma} = 2.76$	$\hat{P}_s = 0.62$ $\tilde{\gamma} = 8.41$
1	150	1000	5.2	3.71	5.49	5.50
2	100	500	6.0	4.28	6.33	6.35
3	250	250	13.5	9.58	14.16	14.22

Note: To convert scale parameter values to metric, multiply by 2.54.

quired to structural stocking guide theory, we can define the augmented information vector $\hat{\boldsymbol{\psi}}_s' = (B_s, N_s, \bar{D}_{q_s}, \hat{P}_s, \hat{D}_{\min_s})$, where $\hat{D}_{\min_s}$ is an estimate of $D_{\min_s}$. In practice, the minimum diameter can be estimated from a walkthrough of the stand under consideration in most situations; its inclusion, therefore, results in minimal additional information required for the model beyond the normal stocking guide parameters. In addition, to minimize notation, it should be understood from the context that the components B_s, N_s, and $\bar{D}_{q_s}$ of $\hat{\boldsymbol{\psi}}_s'$ can also be estimates rather than quantities known without error.

Structural stocking guides

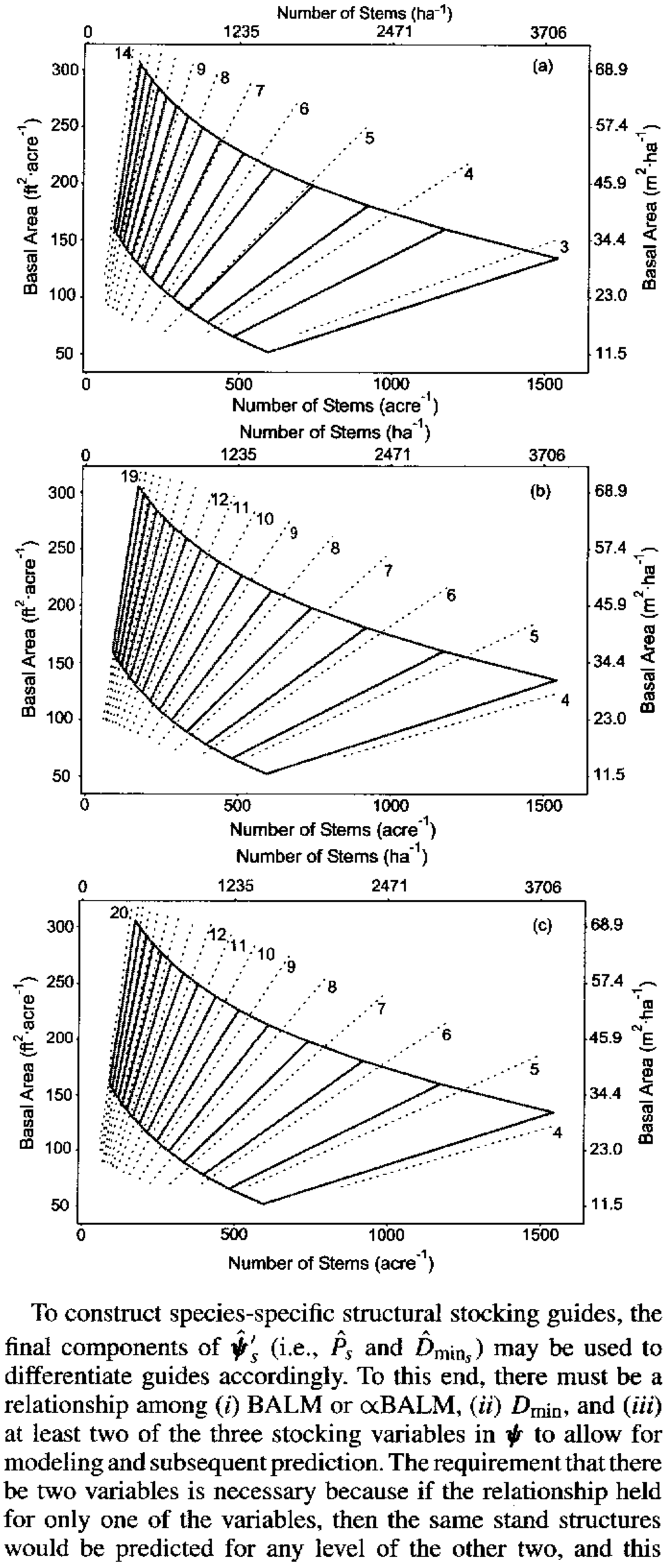
One of the strengths of conventional stocking guides is the useful portrayal of three related variables in an easy-to-understand graphical guide. Because of this familiar structure, it makes some sense to adapt the parameter recovey methods of the previous section to the graphical realm as well. The tact taken in this section is to represent the parameters of the Weibull distribution as contours overlaid on the stocking guide plane. This can be accomplished in a several different ways, two of which are discussed in detail in this section. Alternative methods for representing this information can also be readily envisioned but will not be discussed here.

Perhaps the simplest way to begin is to assume initially that a forester can provide an estimate $\hat{P}$ of ∝BALM for a given stand by some method such as ocular estimation. Based on the discussion in the previous section and the examples in Table 1 and Figure 2, we know for a fixed level of ∝BALM the shape parameter is also fixed and only the scale parameter varies from point to point on the stocking guide. This suggests that one can generate a contoured prototypical structural stocking guide of the scale parameter at every possible level of the shape parameter. In practice, this is done by fixing ∝BALM at a given level and solving the parameter recovery model [4] at a grid of points across the N–B plane. The solutions will have shape parameter estimates $\tilde{\gamma}$ that are all the same, but the estimates $\tilde{\beta}$ will vary from point to point; it is these scale parameter values that can be contoured with many available statistical software packages. Figure 3 illustrates this idea using the same levels of $\hat{P}$ that are found in Table 1 overlaid on top of an eastern white pine stocking guide for New England (Philbrook et al. 1973). Note that the contours for the guides in both Figures 3*b* and 3*c* are very similar. Comparing the shape of the distributions for these levels (Fig. 2), one can readily see that this is because they are both relatively bell-shaped, whereas the distributions at $\hat{P} = 0.83$ are reverse J-shaped. In other words, the form of the distributions change in some nonlinear manner with ∝BALM. Finally, note that it is quite possible to approximate the estimated scale parameter values in Table 1 simply by interpolating from Figure 3.

There are three points that should be made concerning these guides. First, notice that regardless of the level of ∝BALM, the contours for the scale parameter always parallel the lines of constant $\bar{D}_q$ on the guides. This suggests a very strong correlation between $\bar{D}_q$ and $\tilde{\beta}$. Second, these guides are prototypical because, while they embody the essence of incorporating diameter distribution information into the stocking guide, they lack any species or species assemblage specificity. In other words, if a spruce–fir stand and a white pine stand both had estimated $\hat{P} = 0.7$, the contours for the guide in Figure 3*b* would be used for both stands regardless of the where the point s fell on the N–B plane within the respective guides. Third, there is no way to differentiate stands adjusting for $D_{\min}$ with these prototypical guides.

Differentiation by species or stand types is, however, an important component of stocking guides. Just as these different stand types do not share the same A- and B-lines on conventional stocking guides, it seems unlikely that a white pine stand and spruce–fir stand that shared the same stocking parameters (B_s, N_s, $\bar{D}_{q_s}$) would also share the same diameter distribution at point s. Constructing individual structural stocking guides that overlay onto existing guides such as the northern hardwoods (Leak et al. 1987), spruce–fir (Frank and Bjorkbom 1973), white pine (Leak and Lamson 1999; Philbrook et al. 1973; Seymour and Smith 1987), or other guides requires some relationships that will allow customization of the Weibull parameter contours. To illustrate the methods used to accomplish this, a small data set of white pine growth plots from New Hampshire is used. The Hatch plot data set was established in the early 1960s and consisted of permanent plots ranging in size from 1/20 acre to 1/5 acre (0.02–0.8 ha), with one plot per stand throughout New Hampshire. These plots were established in pure, even-aged white pine stands that had no less that 80% white pine by overstory basal area (Barrett and Goldsmith 1973). Approximately 100 plots were originally established, of which only a subset remain today — many having been cut or converted in the intervening years. In the current study, 38 of these Hatch plots that are still in existence were used, with up to 11 remeasurements on each plot. These data are part of the original data set used to establish the A- and B-lines on the original white pine stocking guide (Philbrook 1971; Philbrook et al. 1973).

Fig. 3. Prototypical structural stocking guides of the Weibull scale parameter (dashed lines) for the levels of constant ∝BALM given in Table 1: (*a*) $\hat{P}_s = 0.83$, (*b*) $\hat{P}_s = 0.7$, and (*c*) $\hat{P}_s = 0.62$. The stocking guide over which the contours are laid is for eastern white pine with $\bar{D}_q$ ranging from 4 to 18 inches (10.1–40.6 cm). To convert scale parameter values to metric, multiply by 2.54.



To construct species-specific structural stocking guides, the final components of $\hat{\boldsymbol{\psi}}'_s$ (i.e., $\hat{P}_s$ and $\hat{D}_{\mathrm{min}_s}$) may be used to differentiate guides accordingly. To this end, there must be a relationship among (*i*) BALM or ∝BALM, (*ii*) D_{min}, and (*iii*) at least two of the three stocking variables in $\boldsymbol{\psi}$ to allow for modeling and subsequent prediction. The requirement that there be two variables is necessary because if the relationship held for only one of the variables, then the same stand structures would be predicted for any level of the other two, and this is undoubtedly an unrealistic situation. With the Hatch data, ∝BALM shows weak correlations with individual components of $\boldsymbol{\psi}$ and D_{min}; however, stronger correlations with BALM are evident in these data. Figure 4 presents a plot of the Hatch data with the growth traces of BALM in relation to B over the recorded remeasurements. While a few plots show discernable interruption of the linear trend — possibly due to harvesting activities — the majority of the plots show a strongly consistent linear trend. Similar trends were also found for N, $\bar{D}_q$, and D_{min}, though not quite as strong.

Because of the nature of these data (longitudinal, with probable population (fixed) and sample unit (random) effects), a linear mixed-effects model was determined to be appropriate. Based on guidelines for model selection presented by (Pinheiro and Bates 2000), the following model was chosen:

$$[5] \qquad \mathcal{B}_i = \mathbf{X}_i\boldsymbol{\beta} + \mathbf{Z}_i\mathbf{b}_i + \boldsymbol{\epsilon}_i, \qquad i = 1, \ldots, M$$

where $\mathcal{B}_i$ is the n_i-dimensional (number of observations) response vector, BALM, for the ith plot, with $M = 38$ plots. The fixed effects regressor matrix, $\mathbf{X}_i$, includes variables B, $\bar{D}_q$, and D_{min}. The random effects regressors $\mathbf{Z}_i$ again include B and D_{min}. In this model formulation, the $\boldsymbol{\beta}$ and $\mathbf{b}_i$ correspond to the fixed and random effects, respectively, and the $\boldsymbol{\epsilon}_i \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{\Lambda}_i)$ are the n_i-dimensional within-plot errors, assumed to be independent of the random effects and between plots. In addition, the random effects are assumed $\mathbf{b}_i \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Psi})$.

The model formulation given in [5] does not include intercept terms for either the fixed or random effects components. The no-intercept formulation was judged superior to models that included an intercept based on likelihood ratio tests. In addition, the longitudinal structure of the within-plot measurements constitutes a time series with missing data and unequal remeasurement intervals. An exponential variogram model (Pinheiro and Bates 2000, page 232) with no nugget term was used to model the within-plot correlation structure of the residuals. The final fitted model for the prediction of BALM is

$$[6] \qquad \hat{\mathcal{B}}_s = 0.6603B_s + 1.6547\bar{D}_{q_s} - 3.5891D_{\mathrm{min}_s}$$

where $\hat{\mathcal{B}}_s$ is the predicted value for BALM, adjusted for the stand parameters at point s on the eastern white pine stocking guide and the associated minimum stand DBH. This equation is remarkable in its simplicity and shows a strong relationship between these variables.

The relationship in [6] suggests a straightforward method for constructing a structural stocking guide for eastern white pine. The following steps are applied over a grid of points on the N–B plane:

1. First, for a given point s on the stocking guide with parameters $\boldsymbol{\psi}_s = (B_s, N_s, \bar{D}_{q_s})$ and D_{min_s}, predict $\hat{\mathcal{B}}_s$ from [6].
2. Next, compute $\hat{P}_s = \hat{\mathcal{B}}_s / B_s$, yielding the estimated vector $\hat{\boldsymbol{\psi}}'_s$.
3. Finally, solve the parameter recovery problem [4] at point s using $\hat{\boldsymbol{\psi}}'_s$ and yielding the estimated parameter vector $\tilde{\boldsymbol{\theta}}'_s = (\tilde{\gamma}'_s, \tilde{\beta}'_s)$.

When this procedure is applied to each grid point, the resulting values for $\tilde{\boldsymbol{\theta}}'$ may be individually contoured.

Fig. 4. Growth traces of BALM and B for 38 Hatch plots with up to 11 remeasurements per plot.

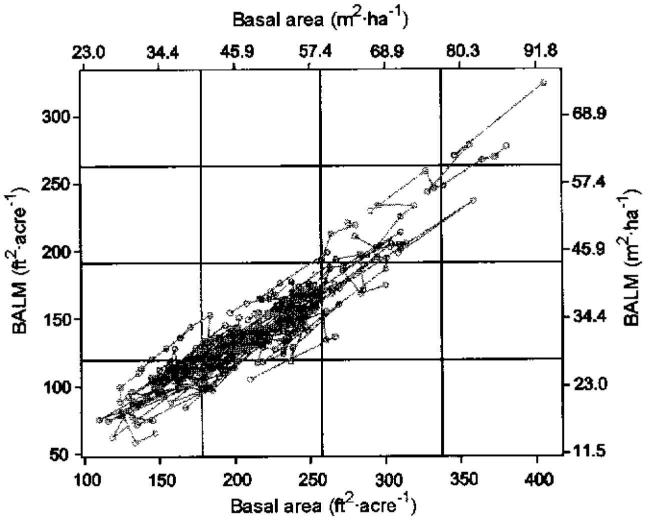
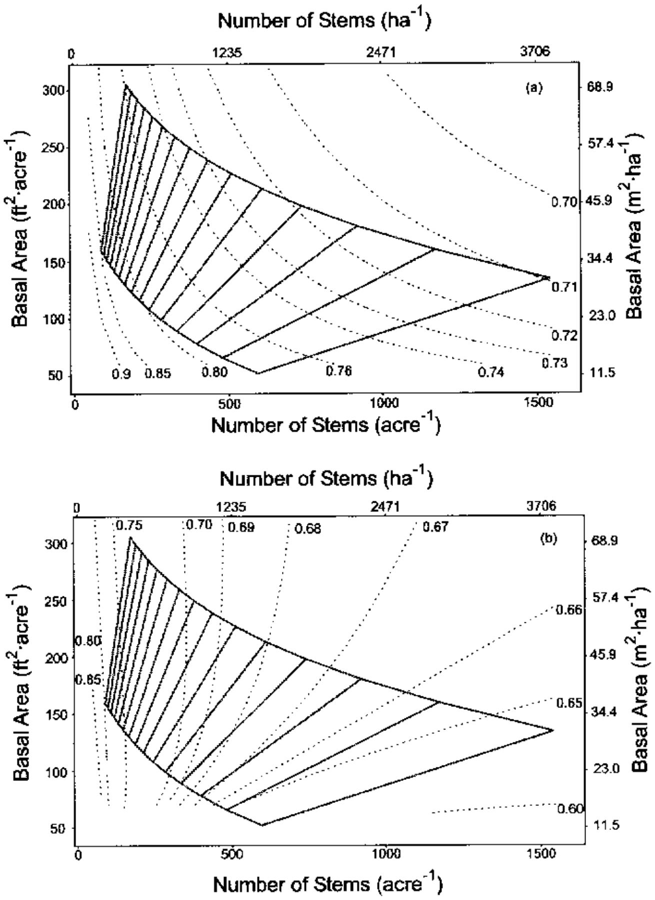


Fig. 5. Interpolated surface of predictions $\hat{P}$ (dashed lines) from model [6] for (*a*) $D_{\min} = 0.0$ in. and (*b*) $D_{\min} = 2.5$ in. with eastern white pine stocking guide.



The above procedure was applied in two ways to illustrate the development of structural stocking guides. In both cases, minimum stand diameters of 0 and 2.5 in. have been used to illustrate the need for considering $D_{\min}$ in the development of these guides. The first set of guides presented in Fig. 5 are not structural stocking guides, but simply show the prediction $\hat{P}$ of $\propto$BALM from eq. 6 overlaid onto the N–B plane. Such a graphical presentation is helpful in envisioning the interaction of stand variables $\boldsymbol{\psi}_s$ and $\propto$BALM, since this is evidently a new measure applied to stocking. The first thing to notice is that the contours in Fig. 5 clearly show differences based on $D_{\min}$. When $D_{\min} = 0$ in., the fully stocked stands generally follow a relatively bell-shaped distribution (c.f. Fig. 2). As we proceed towards the B-line on the stocking guide, the stands become more and more positively skewed, until they actually become reverse J-shaped in the understocked condition. In contrast, the predicted contours for $\propto$BALM on the $D_{\min} = 2.5$ in. guide tend to parallel the lines of constant $\bar{D}_q$ at smaller diameters and depart from this almost paralleling constant number of stems as $\bar{D}_q$ increases. Predicted shapes range from negatively skewed in the lower ranges of $\bar{D}_q$ again to almost reverse J-shaped for stands with largest $\bar{D}_q$ in the extent of the guide. Clearly, the shape of underlying stand distributions, as judged by the surrogate variable $\propto$BALM, change with changing $D_{\min}$ under this paradigm.

While the guides in Fig. 5 are interesting in the sense that they provide a feel for the variation in shape of model distributions over the stocking guide, they lack critical information about distribution scale. The above gridding procedure was again applied at the same levels of $D_{\min}$, this time to the estimated Weibull parameters, and the resulting structural stocking guides for eastern white pine are shown in Figs. 6 and 7. Notice that two views of each guide are generated, one each for the scale and shape parameters. This was done to lessen confusion for the first presentation of such guides. It is entirely possible to combine this information into one chart if desired.

Notice on these structural stocking guides that the scale parameter parallels the lines of equal $\bar{D}_q$, just as in the prototypical guides. Likewise, notice that the predicted shape parameter contours parallel the contours for $\propto$BALM shown in Fig. 5, providing further proof of the fact that $\propto$BALM effectively determines the shapes of the resulting Weibull distributions. To use these guides, one simply determines where the stand of interest $\boldsymbol{\psi}_s$ falls on the guide with the desired value of $D_{\min_s}$ and interpolates both $\tilde{\gamma}'_s$ and $\tilde{\beta}'_s$ from the contours. For example, if the minimum DBH for a stand is $D_{\min} = 0$ in. and $\boldsymbol{\psi}_s = (200, 500, 8.6)$ ($45.9\ \mathrm{m^2 \cdot ha^{-1}}$, $1235.5\ \mathrm{ha^{-1}}$, 21.8 cm), then $\tilde{\boldsymbol{\theta}}'_s$ can be interpolated from Fig. 6 as $\tilde{\gamma}'_s = 2.1$ and $\tilde{\beta}'_s = 8.6$ in. (21.8 cm). Furthermore, the predicted value of $\hat{P}_s = 0.731$ at this point using [6] can be interpolated to a reasonable degree of accuracy from Fig. 5. The actual estimates from solving [4] with the above stand parameters are (2.081, 8.632), a close agreement. These estimated Weibull parameter values found via interpolation from the structural stocking guide, or from solving [4] directly, can then be used with the familiar two-parameter cumulative distribution function to recover the predicted diameter distribution as usual; viz

$$\hat{N}_{sd} = N_s\{\exp[-(d_l/\tilde{\beta}'_s)^{\tilde{\gamma}'_s}] - \exp[-(d_u/\tilde{\beta}'_s)^{\tilde{\gamma}'_s}]\}$$

where d_l and d_u are the lower and upper limits, respectively, for the dth diameter class, and $\hat{N}_{sd}$ is the number of trees in the dth diameter class at structural stocking guide point s.

Fig. 6. A structural stocking guide for eastern white pine for $D_{\min} = 0.0$ in. with (*a*) shape parameter estimates $\tilde{\gamma}_s'$ and (*b*) scale parameter estimates $\tilde{\beta}_s'$ — all scale contours turned to parallel the 4 in. contour at low N (not shown for clarity). To convert scale estimates to metric, multiply by 2.54.

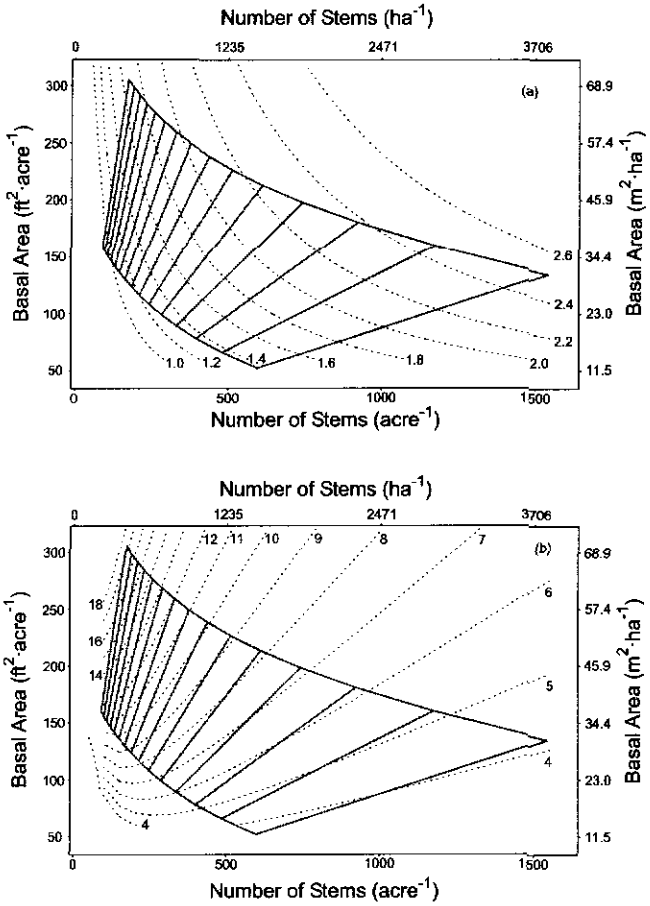
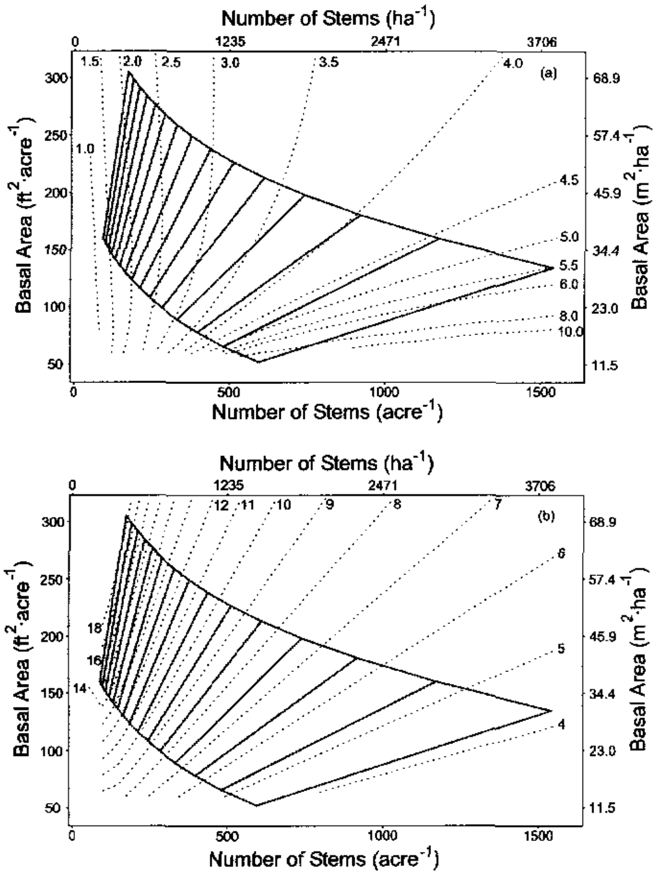


Fig. 7. A structural stocking guide for eastern white pine for $D_{\min} = 2.5$ in. with (*a*) shape parameter estimates $\tilde{\gamma}_s'$ and (*b*) scale parameter estimates $\tilde{\beta}_s'$ — all scale contours turned to parallel the 14 in. contour at low N (not shown for clarity). To convert scale estimates to metric, multiply by 2.54.



In theory, using models [6] and [4] would seem to be the preferred method for predicting a stand structure for eastern white pine based on the structural stocking guide paradigm. However, it appears from the above example and from other tests that little information is lost in the interpolation process by using the graphical approach, providing that the grid spacing is tight enough to allow accurate contouring in constructing the guides. In the creation of the contoured ∝BALM surfaces and the actual structural stocking guides presented in Figs. 5, 6, and 7, gridding ranged from $B = 50$–$320\,\mathrm{ft}^2$ by increments of $10\,\mathrm{ft}^2$, whereas N ranged from 50 to 1550 in increments of 50 trees per acre. Such spacings are evidently adequate to prepare guides for contouring while preserving a reasonable degree of accuracy to the underlying model.

Examples

Unfortunately, it is difficult to design simulations to test a model such as the one presented here. The problem is that our only concept of the stand structure of white pine stands comes from the model and possesses the assumptions embodied therein. In other words, it is not possible to generate, for example, stand DBH distributions at a given point on the stocking guide that represent eastern white pine stands without resorting to the information on the contours in the figures — viz, the structural stocking guide model. This would be employing circular reasoning. For example, suppose we want to generate stand diameter distributions for eastern white pine at a point s with $B_s = 100$ and $N_s = 500$. The only knowledge of the DBH distribution structure (i.e., the Weibull parameters) at that point comes from the contours in Figs 6 and 7 and depends on $D_{\min}$. Therefore, we must rely on data from sampled pure eastern white pine stands to test the model.

There are two aspects of the completed structural stocking guide model framework that can be checked with examples. The first is the notion that ∝BALM is useful in defining the shape of the underlying diameter distribution independently of the subsequent prediction relationship in [6]. Second, distributions recovered using predicted values of BALM, and thus ∝BALM, must also be verified as reasonable. Both models can be compared against known stand data for this exercise. In the following examples, [4] is solved with parameters $\boldsymbol{\psi}_s'$ estimated directly from the stand tables for each stand yielding solution vector $\tilde{\boldsymbol{\theta}}_s$. Then, [4] is again solved, but using $\hat{\boldsymbol{\psi}}_s'$, where $\hat{P}_s$ is predicted using [6]; this yields solution vector $\tilde{\boldsymbol{\theta}}_s'$. These recovered distributions are compared to the two-parameter Weibull

Table 2. Estimated stand and diameter distribution parameters for example stands corresponding to Fig. 8.

	Stand			
	Doe	Hodgeman	Lamson	Mast Yard–A
B (ft^2·acre^{-1})	156.4	190.6	169.3	185.2
N (acre^{-1})	340.1	213.3	260.1	493.8
$\bar{D}_q$ (in.)	9.2	12.8	10.9	8.3
$D_{\min}$ (in.)	0.0	5.0	2.5	0.0
$\propto$BALM	0.81	0.63	0.73	0.74
$\hat{P}$	0.76	0.67	0.71	0.73
$\hat{\boldsymbol{\theta}} = (\hat{\gamma}, \hat{\beta})$	(1.55,8.41)	(3.29,13.20)	(1.92,10.30)	(1.76,8.03)
$\tilde{\boldsymbol{\theta}} = (\tilde{\gamma}, \tilde{\beta})$	(1.15,7.26)	(6.51,13.52)	(2.06,10.99)	(1.92,8.21)
$\tilde{\boldsymbol{\theta}}' = (\tilde{\gamma}', \tilde{\beta}')$	(1.65,8.72)	(3.90,13.59)	(2.14,11.27)	(2.03,8.32)
No. of samples (n)	55	16	15	4

Note: $\propto$BALM is estimated from the stand inventory in this table. To convert scale parameter values to metric, multiply by 2.54.

fitted by maximum likelihood (ML) for each stand, denoted $\hat{\boldsymbol{\theta}} = (\hat{\gamma}, \hat{\beta})$. The ML estimates can be generated because detailed inventory information is available for the example stands.

It may be tempting to use data from individual plots or points to verify the relationships of the structural stocking guide model presented. However, even though the stocking guide A- and B-lines were developed using individual plot data, it must be remembered that the proper application of such guides, with or without structural diameter class information, is with inventories taken over multiple points within relatively homogeneous stands. In the application of SSGs, this admonition becomes even more important because estimated stand tables can vary substantially from point to point within a stand, and the practice of taking enough samples to get a smoother version of the estimated stand table is recommended, especially in nonhomogeneous stands. Available inventory information on pure stands was used, and insofar as possible, an attempt was made to choose stands over a range of the stocking guide.

The results are shown in Table 2, with the stand diameter distributions plotted in Fig. 8. All stands are pure eastern white pine with various mix of other species, but not less than 75% overstory basal area in white pine. The stands were inventoried with either horizontal point sampling (HPS) or fixed radius plot (FRP) sampling. The number of sample units n taken in each stand are shown in Table 2. Where HPS was used, the ML estimates were fitted by combining all points and using the size-biased likelihood approach (Gove 2000, 2003; Van Deusen 1986). For stands inventoried with FRPs, the ML estimates were fitted to the estimated stand table. The program BALANCE (Gove 2003*a*) was used to produce the ML estimates in either case.

At first glance, the results presented in Fig. 8 appear mixed: in Figs. 8*c* and 8*d* the distributions from the stand-based estimate of $\propto$BALM seem to fit well; however, in Figs. 8*a* and 8*b* they do not. The reason for this evidently lies in a point that was mentioned earlier; there can be many different stand distributions that share the same value of $\propto$BALM. The Hodgeman tract (Fig. 8*b*) illustrates this best where the underlying stand diameter distribution is well-behaved, but the estimated distribution based on $\tilde{\boldsymbol{\theta}}_s$ grossly overestimates the peakedness in the distribution. The reason for this is that there is no information in $\propto$BALM when calculated directly from the stand tables about the minimum DBH of 5.5 in. in this stand, a value that is quite significant. Comparing the curve generated from the estimates $\tilde{\boldsymbol{\theta}}'_s$ in the same figure, one could argue that the adjusted prediction of $\hat{P}$ from model [6] leads to even slightly better estimates for this stand than maximum likelihood. Similarly, the Doe Farm in Fig. 8*a* is a slightly bimodal distribution. However, the theory for the use of $\propto$BALM in establishing the shape of the underlying distribution assumes unimodality. Thus, a bias again appears in the estimates $\tilde{\boldsymbol{\theta}}_s$ in this example. The estimates $\tilde{\boldsymbol{\theta}}'_s$ resulting from model-prediction of $\hat{P}$ result in a distribution curve that is virtually indistinguishable from that generated by ML.

The main point to the analysis of the Doe and Hodgeman tracts is that stand estimates of $\propto$BALM may require model adjustment based on the minimum DBH sampled in the stand. Estimates calculated directly from the stand table do not contain any such adjustment and could represent other stand distributions with similar values of N_s and B_s. The adjustments are species-specific, and equations similar to [6] can be empirically fitted to accomplish this adjustment. The values in Table 2 show that the adjustment can work in either direction in the stands presented.

The stands in Fig. 8*c* and 8*d* are comparatively well behaved in both cases. The Lamson Farm (Fig. 8*c*) has a spike of small trees in the 3-in. class. However, this evidently acts like a weighting factor — as if the basal area of these 3-in. trees were spread according to the underlying distribution across the 1- to 3-in. classes — since there is little practical difference in the estimated PR distributions from ML. Lastly, Mast Yard–A presents a very well-behaved, even-aged stand with all diameters measured — such stands are evidently modeled well by the structural stocking guide PR technique.

Discussion

For modeling methods used to develop structural stocking guides in this paper to become useful, they must also hold for

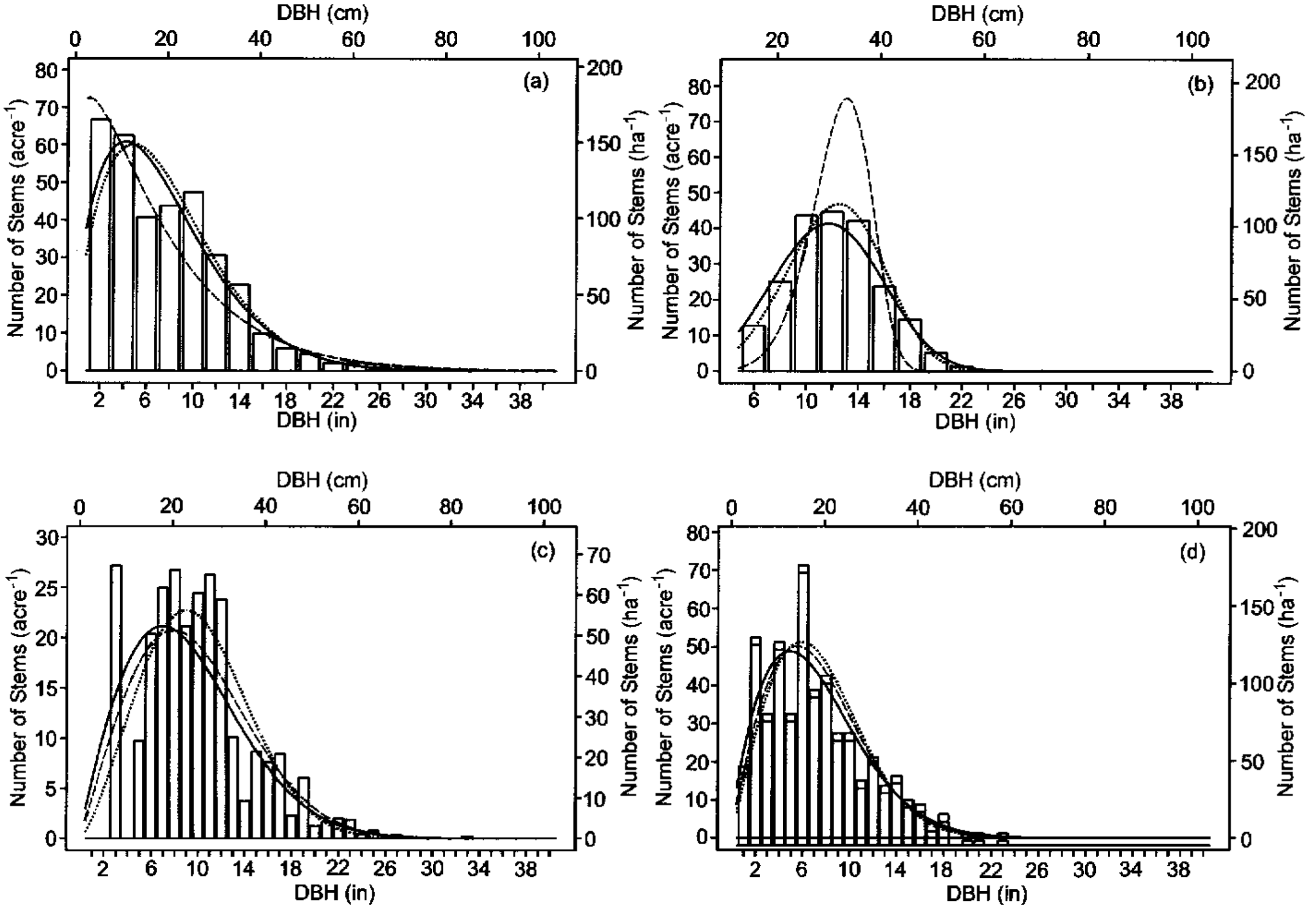


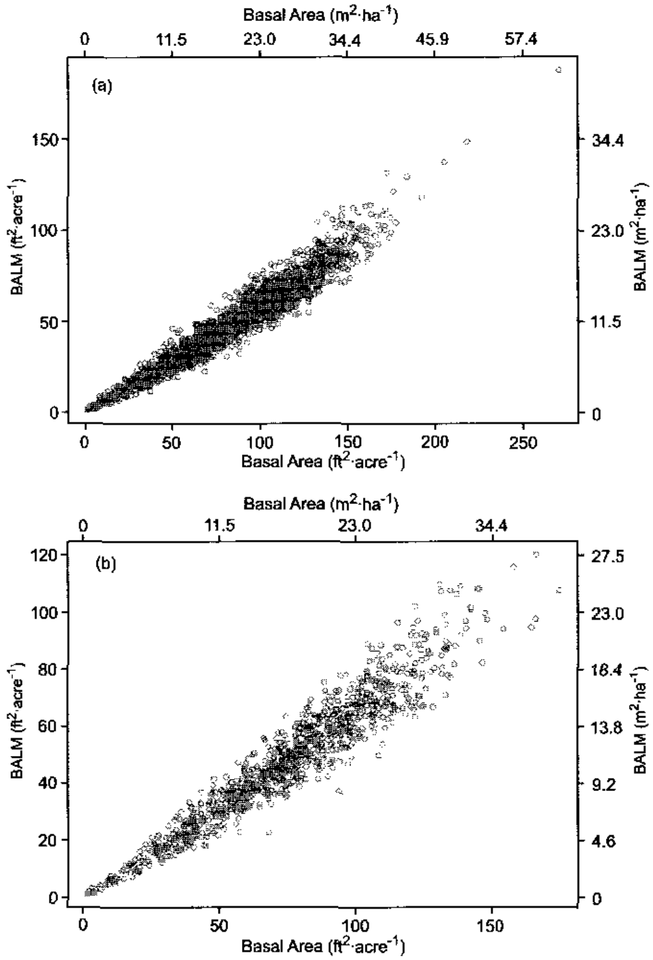
Fig. 8. Example stand diameter distributions with maximum likelihood (solid), stand-based ∝BALM (dashed), and model estimate $\hat{P}$ (dotted) parameter recovery fits for (*a*) Doe Farm, (*b*) Hodgeman, (*c*) Lamson Farm, and (*d*) Mast Yard–A (see Table 2).

other species and species associations found in existing guides. It is clear that if ∝BALM is a useful predictor of diameter distribution shape, then this result is invariant across such lines and the model in [4] holds regardless of species. This can easily be tested by applying [4] to stands with detailed inventory data that allow comparing this parameter recovery model against ML as has been done for eastern white pine here. However, the more uncertain relationship is that of [6] and the question can be posed as to whether it holds for any other species other than white pine or if it is simply a phenomenon of fully stocked pure eastern white pine stands? Figure 9 presents relationships for both the spruce–fir and northern hardwoods habitat types similar to that found in Fig. 4 for eastern white pine. Clearly, the relationships hold for these species assemblages too. Indeed, in both cases, all three variables (B, N, $\bar{D}_q$) were significant in a simple linear regression of BALM on ψ. For spruce–fir, there were $n = 1799$ individual cross-sectional plot observations with an $R^2 = 0.94$; for northern hardwoods, $n = 946$ and $R^2 = 0.92$. Both data sets consisted of managed and unmanaged plots. It can be conjectured, therefore, that the relationship between B and BALM, at least, may be fairly universal for temperate forest species associations. In addition, because the plots for both data sets span the full range of the stocking guide, this undoubtedly contributes to the significance of N and $\bar{D}_q$ in the regressions. Such results are highly desirable because they provide for guides that adjust the prediction of BALM (and thus ∝BALM) over the N–B plane, defining different stand structures throughout.

Given the above results, the structural stocking guides presented in Figs. 6 and 7 should not yet be thought of as the final guide for eastern white pine. The main reason is that the plots used to calibrate eq. 6 do not adequately span all of the portions of the N–B plane that are relevant to the stocking guide, with understocked and smaller diameter ($\bar{D}_q < 6$ in.) stands under-represented in the Hatch data. However, the Hatch white pine data set was chosen to demonstrate the theory of SSG because it provided an example data set that allowed simple illustration of the concepts both numerically and graphically without becoming overwhelmed in data (e.g., Fig. 4).

In addition, pure stands of eastern white pine behave reasonably well with respect to the proposed Weibull parameter recovery eq. 4. It can be shown numerically (Appendix A) that the minimum ∝BALM that is possible with the two-parameter Weibull distribution is approximately 0.575 and that the maximum is approximately 0.999. Distributions within this range effectively span the skewness-kurtosis curve for the Weibull in the Pearson plane. Thus, any stand having a ∝BALM or $\hat{P}_s$ that is below this lower bound can not be fitted with the pa-

Fig. 9. Scatter plot of BALM versus B for the (a) spruce–fir (n = 1799) and (b) northern hardwoods (n = 946) data sets.



rameter recovery model [4] because they are beyond the limit of the Weibull's flexibility. In such cases, another distribution must be sought that is more flexible. An immediate candidate is Johnson's S_B distribution (Johnson 1949). Unfortunately, the added flexibility comes with a price: Johnson's S_B has four parameters that must be estimated, requiring two more estimating equations for [4]. Both the spruce–fir and northern hardwoods data sets had numerous sample plots that fell below the lower limit of ∝BALM for the Weibull. Comparatively, several plots in the Hatch data set had one or more measurements that fell below this lower limit. This presents yet another reason for using model-adjusted estimates of ∝BALM for candidate stands.

The relative inflexibility of the Weibull compared to distributions like S_B should not be misconstrued as a weakness of the structural stocking guide parameter recovery approach in general. One can readily envision a number of estimating equations tied to other stand attributes that can be used to augment [4]. Such equations could take the form of simple, whole-stand prediction equations, such as those used traditionally in growth modeling. Examples of more complex parameter recovery schemes such as these are available in the literature (Lynch and Moser 1986; Murphy and Farrar 1988). In addition, other augmented guides (Goelz 1990; Leary and Standfield 1986; Seymour and Smith 1987) may provide a possible source for underlying model relationships. The generation of structural stocking guides is inherently complicated enough that the simplest approach was chosen to introduce the subject — hence the choice of the eastern white pine guide discussed here.

The latter point on the flexibility of the Weibull is one that must be considered when deciding which distribution to use in [4]. However, it must be stressed again that relationship [6], like the A- and B-lines on the guide, is developed based on individual plots. Thus, the diameter distributions associated with these data may be far more variable than those estimated for a homogenous stand with an appropriate sample support. It is this stand data that is actually used in the final parameter recovery model, and thus this may be somewhat less of a concern. In addition, no attempt was made to stratify the points in Fig. 9 by harvest history. It is possible that those plots that have been cut turn out to be the ones with the most aberrant values of ∝BALM, especially given the fact that the Hatch plots were all uncut in recent history.

The parameter recovery model presented here can also be used for refinement and redefinition of certain existing attributes of stocking guides. For example, in the normal course of stocking guide development, the B-line is constructed by predicting the crown area for the tree of a given mean stand diameter. This area is subsequently factored into the area of one acre (or hectare) to arrive at the number of trees that can fit in an acre when all crowns are touching. Various projected crown shapes are used for this scheme. However, one unreasonable assumption in this technique (note that Seymour and Smith (1987) have provided an alternative to this method) is that of the degenerate diameter distribution: a stand where all trees are exactly the size of the tree of mean stand diameter. With the parameter recovery approach, it is a straightforward matter to define a new mathematical programming model for the B-line stands similar to [4]. This model recovers the diameter distribution at given mean stand diameter while simultaneously determining the associated crown areas that can fit into the proposed acre, thus establishing a specific B-line point (J.H. Gove, unpublished data). Assuming a distribution of tree diameters and associated crown areas is undoubtedly a more realistic approach to the development of B-line forest stocking than the degenerate model normally employed.

Finally, it should be remembered that in practice, the quantities in $\boldsymbol{\psi}_s$ will be estimated from a stand inventory. Thus, the estimates will be associated with some degree of sampling error (possibly other sources of error as well; e.g., measurement, classification, etc.). The estimated sampling error will therefore also affect the model predictions from the structural stocking guides. Ducey and Larson (1997, 1999) have presented methods for assessing the bias that can be attributed to such sources of error from stand inventories as related to various popular density management models.

In summary, this paper has presented a paradigm for associating diameter distributions with traditional stocking guides while requiring the forest manager to have no more information than the usual $\boldsymbol{\psi}$ triplet for their use, other than an estimate of the minimum tree DBH. These guides lend themselves both to graphical presentation in the traditional manner and to incorporation into computer packages that can easily solve the equations in [4] and [6] — the heart of the system.

Acknowledgements

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Appendix A. Minimizing and maximizing ∝BALM.

The two-parameter Weibull distribution can take on a number of shapes, but it is limited in its flexibility. Because the shape of the distribution is related to ∝BALM, the equation for $\propto\text{BALM}_{\mathcal{B}}$ can be treated as an objective function to be either minimized or maximized at any point s on the stocking guide; viz

$$\text{[A.1]} \quad \underset{\{\gamma, \beta\}}{\text{Min or Max}} \propto\text{BALM}_{\mathcal{B}} = 1 - \frac{\gamma(k_2, (\bar{D}_q/\beta)^{\gamma})}{\Gamma(k_2)}$$

$$\text{St}: \quad 0.1 \leq \gamma \leq 100.0$$

$$0.1 \leq \beta \leq 200.0$$

$$B_s - B^{\mu} = 0$$

where the first two constraints simply keep the Weibull parameters from becoming ridiculous, and the last constraint fixes the minimization or maximization to a specific point s on the stocking guide using [2].

Solving [A.1] at several points will give very similar answers but does not quite give the global minimum or maximum ∝BALM for the two-parameter Weibull because of the extra constraint from [2]. The above nonlinear program can be easily made independent of the stocking guide by dropping the last constraint and making the substitution used earlier for the quadratic mean stand diameter (i.e., $\bar{D}_q^2 = \beta^2\Gamma(k_2)$) in the objective function; viz

$$\text{[A.2]} \quad \underset{\{\gamma, \beta\}}{\text{Min or Max}} \quad 1 - \frac{\gamma\left(k_2, \Gamma(k_2)^{\gamma/2}\right)}{\Gamma(k_2)}$$

$$\text{St}: \quad 0.1 \leq \gamma \leq 100.0$$

$$0.1 \leq \beta \leq 200.0$$

Solving [A.2] will produce the minimum and maximum values for ∝BALM given earlier while still keeping $\boldsymbol{\theta}$ reasonable.