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Moment and maximum likelihood estimators for Weibull distributions under length- and area-biased sampling

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Many of the most popular sampling schemes used in forestry are probability proportional to size methods. These methods are also referred to as size-biased because sampling is actually from a weighted form of the underlying population distribution. Length- and area-biased sampling are special cases of size-biased sampling where the probability weighting comes from a lineal or areal function of the random variable of interest, respectively. Often, interest is in estimating a parametric probability density of the data. In forestry, the Weibull function has been used extensively for such purposes. Estimating equations for method of moments and maximum likelihood for two- and three-parameter Weibull distributions are presented. Fitting is illustrated with an example from an area-biased angle-gauge sample of standing trees in a woodlot. Finally, some specific points concerning the form of the size-biased densities are reported.

Keywords: horizontal point sampling, point relascope sampling, probability proportional to size, size-biased distributions, weighted distributions

1. Introduction

Size-biased distributions arise naturally from many probability sampling designs used in forestry and related fields. The most useful of these designs can be conveniently categorized as length- and area-biased sampling designs. For length-biased methods, sampling is with probability proportional to some lineal measure; e.g., piece length or diameter. With area-biased designs, individuals are selected into the sample with probability proportional to some areal attribute, the most widely known example is tree basal area. For example, length-biased samples arise from the line intersect (LIS) (Kaiser, 1983) and transect relascope (TRS) (Ståhl, 1998) methods for sampling down coarse woody debris (CWD), and from horizontal line samples (HLS) of standing trees (Grosenbaugh, 1958). In LIS and TRS, piece length is the operative random variable, whereas in HLS it is tree diameter. Similarly, area-biased samples arise from horizontal

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point sampling (HPS) (Grosenbaugh, 1958) where trees are selected with probability proportional to their basal areas, and point relascope sampling (PRS) (Gove *et al.*, 1999) which selects down pieces of CWD with probability proportional to their squared length. In all methods except LIS, individuals are selected with the aid of an angle gauge, which effectively serves to distribute the attributes of interest over a larger area, thus increasing the chances of being selected by a randomly chosen point (Valentine *et al.*, 2001). In LIS, the line itself serves the same function as the angle gauge by creating a larger inclusion area for a downed log than the log itself.

It is often the case that one desires to fit a known probability distribution to sample data. Under equal probability sampling this is straightforward and moment or maximum likelihood estimators have been published for a wide variety of distributions. Let X be the random variable of interest such that $X \sim f(x;\theta)$, then in the equal probability case, one would desire estimates of the unknown parameters θ. However, under size-biased schemes, the probability of sampling an individual is proportional to X^α, $\alpha = 1, 2$ for length- and area-biased sampling, respectively. Therefore, the correct density is of the form (Patil and Ord, 1976; Patil, 1981)

$$f_\alpha^*(x;\theta) = \frac{x^\alpha f(x;\theta)}{\int x^\alpha f(x;\theta)\,dx}, \tag{1}$$

where the denominator—the αth raw moment of $f(x;\theta)$—serves as a normalizing constant for the size-biased density and we write $X_\alpha^* \sim f_\alpha^*(x;\theta)$. Clearly the equal probability moment and likelihood equations do not apply in samples arising from length- or area-biased data because of the unequal weighting. In forestry, Van Deusen (1986) was the first to recognize this with regard to HPS. Later, Lappi and Bailey (1987) showed how the same principles applied to diameter increment arising from HP samples. Gove and Patil (1998) applied size-biased distributions in a pure modeling scenario to the basal area-diameter distribution while Gove (2000, 2003a) showed the consequences of using equal probability methods when size-biased estimation techniques were clearly called for.

Introduced nearly three decades ago to forestry (Bailey and Dell, 1973) the Weibull probability density has become widely used as a diameter distribution model. For example, the Weibull played a major role in the development of parameter prediction and parameter recovery methods (Hyink and Moser, 1983) used in the modeling of forest growth. Other forestry-related uses of the Weibull include applications as varied as precipitation models (Duan *et al.*, 1998) and fire recurrence (Polakow and Dunne, 1999) modeling. In this study, both moment and maximum likelihood (ML) equations are presented for parameter estimation of Weibull distributions arising from length- and area-biased samples.

2. Weibull distributions

The two- and three-parameter Weibull distributions differ only in the inclusion of a location parameter for the three-parameter version. The pdfs for the two- and three-parameter case are given as

$$f(x;\theta) = \left(\frac{\gamma}{\beta}\right)\left(\frac{x}{\beta}\right)^{\gamma-1} e^{-(x/\beta)^\gamma}, \quad x > 0$$

$$f(x;\theta) = \left(\frac{\gamma}{\beta}\right)\left(\frac{x-\xi}{\beta}\right)^{\gamma-1} e^{-((x-\xi)/\beta)^\gamma}, \quad x > \xi,$$

with $\theta = (\gamma, \beta)'$ and $\theta = (\gamma, \beta, \xi)'$, respectively. The unknown parameters $\gamma > 0$, $\beta > 0$ and $\xi > 0$ are the shape, scale and location parameters to be estimated for a given sample of data.

Let the αth raw moment for $f(x;\theta)$ be defined as $\mu'_\alpha = \int x^\alpha f(x;\theta)\,dx$. In the two-parameter case, the moments are always given by the simple relation $\mu'_\alpha = \beta^\alpha \Gamma_\alpha$, where $\Gamma_\alpha = \Gamma(\alpha/\gamma + 1)$ and $\Gamma(k) = \int_0^\infty x^{k-1} e^{-x}\,dx$, $k > 0$, the gamma function. However, the form of the raw moments for the three parameter case varies somewhat according to the integer value of α. For the sake of exposition, let X be two-parameter Weibull with $E[X^\alpha] = \beta^\alpha \Gamma_\alpha$, then $Y = X + \xi$ is three-parameter Weibull and the successive raw moments can be found from $\mu'_\alpha = E[Y^\alpha] = E[(X+\xi)^\alpha]$. Applying the binomial theorem to expand the argument of the expectation yields

$$\begin{aligned}\mu'_\alpha &= E[X^\alpha] + \binom{\alpha}{1} E[X^{\alpha-1}]\xi + \binom{\alpha}{2} E[X^{\alpha-2}]\xi^2 + \cdots + \xi^\alpha \\ &= \beta^\alpha\Gamma_\alpha + \binom{\alpha}{1}\beta^{\alpha-1}\Gamma_{\alpha-1}\xi + \binom{\alpha}{2}\beta^{\alpha-2}\Gamma_{\alpha-2}\xi^2 + \cdots + \xi^\alpha. \qquad (2)\end{aligned}$$

Now, allowing μ'_α to represent the raw moments for the two- and three-parameter Weibull as appropriate, it follows directly from (1) and the above results that the size-biased versions of the two- and three-parameter Weibulls are

$$f^*_\alpha(x;\theta) = (\mu'_\alpha)^{-1} x^\alpha \left(\frac{\gamma}{\beta}\right)\left(\frac{x}{\beta}\right)^{\gamma-1} e^{-(x/\beta)^\gamma}, \quad x > 0$$

$$f^*_\alpha(x;\theta) = (\mu'_\alpha)^{-1} x^\alpha \left(\frac{\gamma}{\beta}\right)\left(\frac{x-\xi}{\beta}\right)^{\gamma-1} e^{-((x-\xi)/\beta)^\gamma}, \quad x > \xi$$

respectively, with the same restrictions on the parameters as for the equal probability pdfs.

3. Moment estimation

The moment equations under size-biased sampling require the raw moments of the size-biased distribution. These moments are simple ratios of the moments of the equal probability forms; define $\mu^{*\prime}_{\alpha,\zeta}$ as the ζth raw moment of the size-biased distribution of order α. Then

$$\begin{aligned}\mu^{*\prime}_{\alpha,\zeta} &= \int x^\zeta f^*_\alpha(x;\theta)\,dx \\ &= \frac{\mu'_{\alpha+\zeta}}{\mu'_\alpha}.\end{aligned}$$

For the two-parameter Weibull, it follows that the raw moments of the size-biased distribution are of the form

$$\mu^{*\prime}_{\alpha,\zeta} = \beta^{\zeta}\frac{\Gamma_{\alpha+\zeta}}{\Gamma_{\alpha}}.$$

For estimation purposes, the first two raw moments of the size-biased two-parameter Weibull can be set equal to the sample moments, the solution of which requires solving a set of two equations simultaneously. Alternatively, a modified method of moments may be preferred. It is a simple matter to calculate the mean and variance of the sample data. From these, the coefficient of variation CV may also be found directly. Modified moment equations can be developed using the first moment and the coefficient of variation; this scheme may be preferable because there is only one equation to solve for one unknown, simplifying estimation as in the equal probability case (Cohen, 1965). The variance of a size-biased random variable of order α is given as usual

$$\mathrm{Var}(x^*_\alpha) = \mu^{*\prime}_{\alpha,2} - \left(\mu^{*\prime}_{\alpha,1}\right)^2,$$

which, for the two-parameter Weibull becomes

$$= \beta^2\left(\frac{\Gamma_{\alpha+2}}{\Gamma_\alpha} - \frac{\Gamma^2_{\alpha+1}}{\Gamma^2_\alpha}\right).$$

The coefficient of variation is defined as the square-root of the variance divided by the mean. In general, the coefficient of variation for the size-biased distribution of order α is

$$\mathfrak{C}^*_\alpha = \frac{\sqrt{\mathrm{Var}(x^*_\alpha)}}{\mu^{*\prime}_{\alpha,1}}.$$

Therefore, after substitution of terms for the two-parameter Weibull, we have

$$\mathfrak{C}^*_\alpha = \Gamma_\alpha\Gamma^{-1}_{\alpha+1}\sqrt{\frac{\Gamma_{\alpha+2}}{\Gamma_\alpha} - \frac{\Gamma^2_{\alpha+1}}{\Gamma^2_\alpha}}.$$

It follows that the modified moment equations for the two-parameter size-biased Weibull distribution of order α are

$$\tilde{\mathfrak{C}}^*_\alpha = CV, \tag{3}$$

$$\tilde{\beta} = \frac{\bar{x}\tilde{\Gamma}_\alpha}{\tilde{\Gamma}_{\alpha+1}}, \tag{4}$$

where the first Equation (3) is solved iteratively for the shape parameter estimate $\tilde{\gamma}$, which is then substituted into the Equation (4) providing the scale parameter estimate $\tilde{\beta}$. Equation (4) derives from equating the sample mean $(\bar{x})$ to the first raw moment $\mu^{*\prime}_{\alpha,1}$.

The moment estimators for the size-biased three-parameter Weibull rely on the relationship developed in (2). The addition of the location parameter adds a level of complexity to the three-parameter form that was not found in the previous development because it requires having separate equations for each size-biased order α. In addition, the formula for the variance is straightforward, but, especially for $\alpha = 2$, the formula for the coefficient of variation becomes overly complex. Alternatively, it would be tempting to follow a modified moment estimation scheme as in Cohen *et al.* (1984) using the mean, variance and first order statistic moments. However, the distribution of order statistics for the size-biased form is intractable. Thus, a simple scheme based solely on the first three moments of the size-biased distribution has been adopted here. In this scheme, we again make use of the relationship for $\mu^{*\prime}_{\alpha,\zeta}$; *viz.*,

$$\mu^{*\prime}_{\alpha,1} = \frac{\mu'_{\alpha+1}}{\mu'_{\alpha}}, \qquad \mu^{*\prime}_{\alpha,2} = \frac{\mu'_{\alpha+2}}{\mu'_{\alpha}}, \qquad \mu^{*\prime}_{\alpha,3} = \frac{\mu'_{\alpha+3}}{\mu'_{\alpha}}. \tag{5}$$

Since $\alpha = 1$ or 2 and $\zeta = 1, \ldots, 3$ in (5), the first five non-central moments $\mu'_1, \mu'_2, \ldots, \mu'_5$ of the three-parameter Weibull are required for the $\mu^{*\prime}_{\alpha,\zeta}$ in the size-biased equations. Substituting in for μ'_{α} and $\mu'_{\alpha+\zeta}$ from (2) with the appropriate value of α and ζ completes the equations. For example, for area-biased samples, the first size-biased moment $\mu^{*\prime}_{2,1}$ is

$$\frac{\mu'_3}{\mu'_2} = \frac{\beta^3\Gamma_3 + 3\xi\beta^2\Gamma_2 + 3\xi^2\beta\Gamma_1 + \xi^3}{\beta^2\Gamma_2 + 2\xi\beta\Gamma_1 + \xi^2}.$$

The solution is found by setting each of the three raw moments equal to the corresponding sample moments and solving the system simultaneously for the three unknown parameter values.

4. Maximum likelihood estimation

Maximum likelihood estimation for size-biased distributions of the form considered here also follows directly from the equal probability case. In general, the log likelihood for the size-biased pdf of the form (1) is

$$\ln \mathscr{L}^* = \alpha \sum_{i=1}^{n} \ln x_i + \sum_{i=1}^{n} \ln f(x_i; \theta) - n \ln \mu'_{\alpha}.$$

As pointed out by Van Deusen (1986), the first term is a constant and may be dropped if desired, the second term is the usual (equal probability) log-likelihood, $\ln \mathscr{L}$, and the third term is a ''correction'' term accounting for the fact the observations were not drawn with equal probability.

Rather than numerically maximizing $\ln \mathscr{L}^*$ directly, it is often more useful to have first- and second-order derivative information for Newton-type algorithms and for variance estimation via the Hessian. The reader is referred to Gove (2000) for the first-order partial derivative equations for the size-biased two-parameter Weibull. Similar equations can be derived for the size-biased three-parameter Weibull. However, as in the case of the moment estimators, the form of the derivative equations again depends on the size-biased order α. The gradient equations are

$$\frac{\partial \ln \mathcal{L}^*}{\partial \gamma} = \beta^{-\gamma}\left\{\ln\beta\sum_{i=1}^{n}(x_i-\xi)^\gamma - \sum_{i=1}^{n}(x_i-\xi)^\gamma \ln(x_i-\xi)\right\}$$
$$+ n(\gamma^{-1} - \ln\beta) + \sum_{i=1}^{n}\ln(x_i-\xi) - n\rho_\gamma(\alpha)$$
$$\frac{\partial \ln \mathcal{L}^*}{\partial \beta} = \frac{\gamma}{\beta^{\gamma+1}}\sum_{i=1}^{n}(x_i-\xi)^\gamma - \frac{n\gamma}{\beta} - n\rho_\beta(\alpha)$$
$$\frac{\partial \ln \mathcal{L}^*}{\partial \xi} = \frac{\gamma}{\beta^\gamma}\sum_{i=1}^{n}(x_i-\xi)^{\gamma-1} - (\gamma-1)\sum_{i=1}^{n}(x_i-\xi)^{-1} - n\rho_\xi(\alpha),$$

where

$$\rho_\gamma(1) = -\frac{\beta\psi_1\Gamma_1}{\gamma^2\mu_1'}, \qquad \rho_\gamma(2) = -\frac{2\beta}{\gamma^2\mu_2'}(\beta\psi_2\Gamma_2 + \xi\psi_1\Gamma_1),$$
$$\rho_\beta(1) = \frac{\Gamma_1}{\mu_1'}, \qquad \rho_\beta(2) = \frac{2}{\mu_2'}(\beta\Gamma_2 + \xi\Gamma_1),$$
$$\rho_\xi(1) = (\mu_1')^{-1}, \qquad \rho_\xi(2) = \frac{2\mu_1'}{\mu_2'},$$

and $\psi_\alpha = \psi(\alpha/\gamma + 1)$ is the digamma function (ψ) (Abramowitz and Stegun, 1964, p. 258) indexed to the size-biased order α. These equations can be combined with second order information (Appendix) and solved for the unknown parameter values by Newton–Raphson iteration using the moment estimates as starting values.

5. Example: horizontal point sampling

Recently, the methods discussed herein for size-biased Weibull estimation have been incorporated into a graphical user interface-based program for fitting diameter distributions to forestry data (Gove, 2003b). A two-stage estimation scheme that seems to work well in many cases is to estimate parameters in the first stage using the moment equations, yielding estimates $\tilde{\theta}$. In the second stage, the moment estimates are used as starting values for maximum likelihood yielding estimates $\hat{\theta}$. This scheme has been suggested in equal probability estimation of Weibull parameters (Cohen, 1965) and is employed in the following example.

The data used in this example are from a forest inventory on a 72 acre parcel of the Mont Vernon New Hampshire town forest tract. The forest is composed of mixed hardwoods and eastern white pine (*Pinus strobus* L.) and has an average basal area of approximately 150 ft^2 per acre. In the course of the field phase of the inventory, 46 prism points were sampled with a 20 basal area factor prism. Data from the tally on the 46 points were lumped yielding a total of 359 sample trees for area-biased Weibull parameter estimation.

Two- and three-parameter area-biased Weibull distributions were fitted to the Mont Vernon data using the methods described above. In both cases, moment estimates were computed first as starting values for MLE. The results are presented in Table 1 along with the Akaike Information Criterion (AIC). In all cases, both moment and ML, the estimates converged quickly to reliable solutions. The remarkable correspondence between the

Table 1. Parameter estimates for Mont Vernon inventory.

	Parameter Estimates			
Estimation Method	γ	β	ξ	AIC
Moments	1.492	7.676	—	
	1.391	6.701	1.704	
Maximum Likelihood	1.506	7.753	—	2233.67
	1.388	6.664	1.773	2233.77

$\text{AIC} = -2\ln \mathfrak{L}^* + 2K$, where K is the number of Weibull parameters.

moment and ML estimates should confirm the soundness of the estimating equations. Corresponding graphical results of the ML fits are shown in Fig. 1. This figure presents both the two- and three-parameter fits (dashed). In addition, it presents the equal probability Weibull densities using the size-biased parameter estimates. The equal probability densities estimate the underlying population distribution from which the sample was drawn. Note that the underlying sample of 359 trees forms a relatively nice, unimodal distribution in this example as shown by the histogram.

6. Discussion and conclusions

While the results from the example in the preceding section with regard to parameter estimation are encouraging, there are a few points worth mentioning with regard to the size-biased Weibull distributions discussed here. First, size-biased three-parameter Weibull distributions can take on a range of shapes, some of which are not found in the associated equal probability Weibull. Fig. 2 presents a set of graphs for the area-biased three parameter Weibull for illustration. Each row in Fig. 2 corresponds to a different value of the shape parameter, ranging from 0.7–2.0, while the columns show the effect of increasing the scale parameter by a factor of two. The value of the location parameter ranges from 0–6 in steps of two to facilitate illustration of the following points.

For shape parameters less than $\gamma = 1$, several different shapes are possible ranging from "L-" to "h-" to almost "n-"shaped, in all cases with large positive skewness. The pdfs in Fig. 2a, with $\gamma = 0.7$ and $\xi > 0$ illustrate intermediate shapes within this range. This contrasts with the equal probability Weibull, which is reverse "J-"shaped for $\gamma \leq 1$. For the sake of clarity in Fig. 2a, the pdfs are plotted beginning at $\xi + 0.01$ rather than at ξ to more clearly illustrate the shape without the straight vertical intercept line that would appear otherwise.

Similarly, the case where the shape parameter equals 1 in Fig. 2b presents another anomaly that might not be evident from the formulæ alone: the pdfs with non-zero location appear to resemble truncated densities. In panel b, the densities have been plotted starting at ξ this time and the vertical line shows clearly the beginning point of each density. Panel *c* also shows this, but as the shape parameter increases, this phenomenon is less pronounced. Indeed, it disappears completely in panel *d* for $\beta = 3$.

Comparison of the panels in Fig. 2 shows the interplay between the different parameters. As the shape parameter increases (rows), the "truncation effect" is lessened. The same

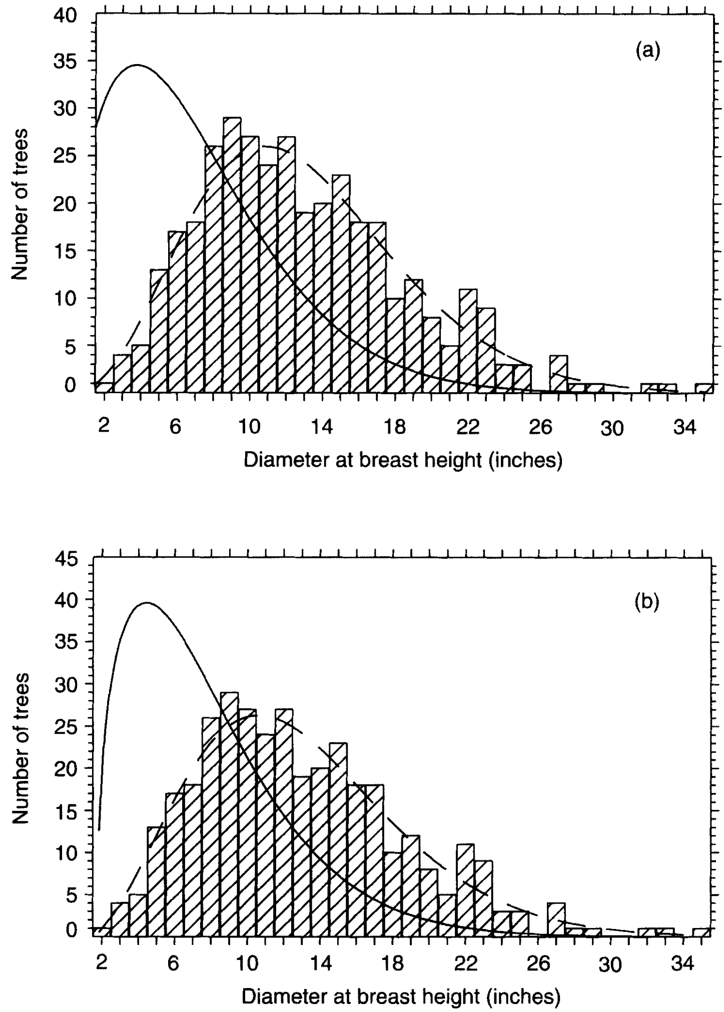

Figure 1. Maximum likelihood fits of area-biased pdfs (dashed) for the Mount Vernon tally (histograms) with underlying population Weibull estimate (solid): (a) two-parameter Weibull, (b) three-parameter Weibull (see Table 1 for coefficients).

can be said for the scale (columns) parameter. For a given set of shape and scale parameters, however, the truncation effect increases as the location parameter increases. This makes intuitive sense: Since the density must integrate to one, the curve must shift upwards to accommodate the extra area.

Table 2 provides more detail into the truncation effect phenomenon. These data correspond to the pdfs in the first column of Fig. 2b with $\gamma = 1$, $\beta = 3$. Here, the components of the area-biased density have been evaluated separately at $x = \xi + 0.01$, just after the intercept, assuring positive probability density. Columns two and three of Table 2 give the numerator components of $f_2^*(x; \theta)$, while the last column gives the denominator, μ_2'. The ratio of the two numerator components is given in column four and

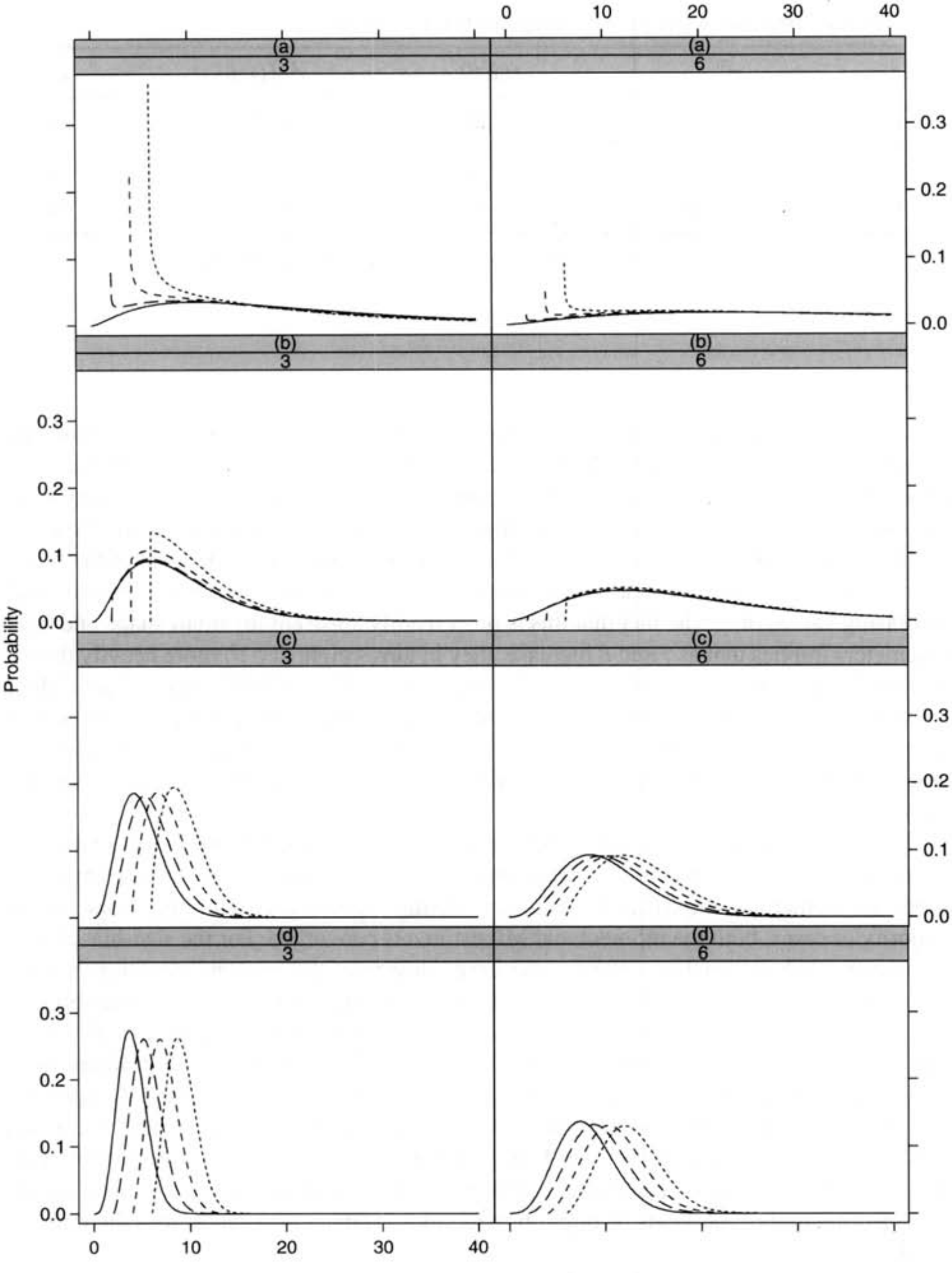

Figure 2. Area-biased three-parameter Weibull distributions with columns $\beta = 3, 6$ and rows: (a) $\gamma = 0.7$, (b) $\gamma = 1.0$, (c) $\gamma = 1.5$, (d) $\gamma = 2$; $\xi = 0$ (solid), 2 (long dash), 4 (short dash), 6 (dotted) for all pdfs.

Table 2. Size-biased pdf ($\alpha = 2$) components for $\gamma = 1, \beta = 3$.

ξ	x^{2a}	$f(x;\theta)$	$x^2/f(x;\theta)$	μ_2'
0	0.0001	0.3322	0.0003	18
2	4.0401	0.3322	12.2	34
4	16.0801	0.3322	48.4	58
6	36.1201	0.3322	108.7	90

[a] $x = \xi + 0.01$, a small amount is added to ξ to make the density non-zero.

corresponds to a relative weighting of the two to facilitate comparison. Note from the definition of the three-parameter Weibull density, that $f(x;\theta)$ is always evaluated at $x - \xi$ rather than x when $\xi > 0$. However, the x^α component is never shifted, since its role is one of generating a moment in the size-biased paradigm. Thus, in any comparison of the two close to ξ, the x^α term will dominate, producing the truncation effect. Perusal of the fourth column in Table 2 verifies that this is indeed the case, and that this trend increases with increasing values of ξ. The fact that this is most readily apparent for small shape and scale parameters implies that as γ and β increase, they in turn weight $f(x;\theta)$ more heavily than x^α and the densities begin at zero, showing no truncation effect. While it may be possible to iteratively solve for the unknown parameters at the point where the pdf equals zero in such cases, it is probably more beneficial to have an intuitive understanding of the phenomenon described above and when to expect it, than trying to find any explicit case where the truncation effect disappears.

Extensive simulations have previously been presented for the ML estimators in the two-parameter case (Gove, 2000). The remaining estimators presented in this paper have been tested on numerous empirical distributions arising from forest sampling work. In the majority of cases, both the moment and ML estimates converged. For the size-biased two-parameter Weibulls, this was always the case. However, the moment equations for the size-biased three-parameter Weibull tended not to converge in some cases. This normally occurred when there was some hint of bimodality in the data, or in the case of small sample size coupled with positively skewed shape parameter less than two. The problem in all cases was evidently that the location parameter is poorly estimated. The ML equations appear to be more robust and normally converge if reasonably good starting values are provided in the three-parameter case. This would suggest that if a more robust form for the three-parameter moment equations could be found, possibly along the lines of the modified method of moments, it might provide more stable equations across the range of legal parameter values.

Appendix

The following equations define the Hessian matrix of second-order information for the size-biased three-parameter Weibull.

$$\frac{\partial^2 \ln \mathscr{L}^*}{\partial\gamma^2} = \beta^{-\gamma}\left\{(\ln\beta)^2\sum_{i=1}^{n}(x_i-\xi)^\gamma - \sum_{i=1}^{n}(x_i-\xi)^\gamma(\ln(x_i-\xi))^2\right\} - \frac{n}{\gamma^2} - n\rho_{\gamma\gamma}(\alpha)$$

$$\frac{\partial^2 \ln \mathscr{L}^*}{\partial\beta^2} = \frac{n\gamma}{\beta^2} - \frac{\gamma(\gamma+1)}{\beta^{\gamma+2}}\sum_{i=1}^{n}(x_i-\xi)^\gamma - n\rho_{\beta\beta}(\alpha)$$

$$\frac{\partial^2 \ln \mathscr{L}^*}{\partial\xi^2} = (1-\gamma)\left\{\frac{\gamma}{\beta^\gamma}\sum_{i=1}^{n}(x_i-\xi)^{\gamma-2} + \sum_{i=1}^{n}(x_i-\xi)^{-2}\right\} - n\rho_{\xi\xi}(\alpha)$$

$$\frac{\partial^2 \ln \mathscr{L}^*}{\partial\gamma\partial\beta} = \beta^{-(\gamma+1)}\left\{(1-\gamma\ln\beta)\sum_{i=1}^{n}(x_i-\xi)^\gamma + \gamma\sum_{i=1}^{n}(x_i-\xi)^\gamma\ln(x_i-\xi)\right\} - \frac{n}{\beta} - n\rho_{\gamma\beta}(\alpha)$$

$$\frac{\partial^2 \ln \mathscr{L}^*}{\partial\gamma\partial\xi} = \beta^{-\gamma}\left\{(1-\gamma\ln\beta)\sum_{i=1}^{n}(x_i-\xi)^{\gamma-1} + \gamma\sum_{i=1}^{n}(x_i-\xi)^{\gamma-1}\ln(x_i-\xi)\right\} - \sum_{i=1}^{n}(x_i-\xi)^{-1} - n\rho_{\gamma\xi}(\alpha)$$

$$\frac{\partial^2 \ln \mathscr{L}^*}{\partial\beta\partial\xi} = -\frac{\gamma^2}{\beta^{\gamma+1}}\sum_{i=1}^{n}(x_i-\xi)^{\gamma-1} - n\rho_{\beta\xi}(\alpha),$$

where

$$\rho_{\gamma\gamma}(1) = \frac{\beta\mu_1'\Gamma_1\lambda_1}{\gamma^4}\left\{\tilde{\psi}_1 + \psi_1\left(2\gamma - \frac{\beta\psi_1\Gamma_1}{\mu_1'}\right)\right\}$$

$$\rho_{\gamma\gamma}(2) = \frac{\beta\lambda_2}{\gamma^4}\left\{\mu_2'(2\beta\Gamma_2\tilde{\psi}_2 - \xi\Gamma_1\tilde{\psi}_1) + 2\nu_1(\gamma\mu_2' - \beta\nu_1)\right\}$$

$$\rho_{\beta\beta}(1) = -\Gamma_1^2\lambda_1$$

$$\rho_{\beta\beta}(2) = \lambda_2(\mu_2'\Gamma_2 - 2\nu_2^2)$$

$$\rho_{\xi\xi}(1) = -\lambda_1$$

$$\rho_{\xi\xi}(2) = \lambda_2(\mu_2' - 2\mu_1')$$

$$\rho_{\gamma\beta}(1) = \frac{\lambda_1}{\gamma^2}(\beta\psi_1\Gamma_1^2 - \mu_1'\psi_1\Gamma_1)$$

$$\rho_{\gamma\beta}(2) = \frac{-\lambda_2}{\gamma^2}\{\mu_2'(2\beta\psi_2\Gamma_2 + \xi\psi_1\Gamma_1) - 2\beta\nu_1\nu_2\}$$

$$\rho_{\gamma\xi}(1) = \frac{\beta\psi_1\Gamma_1\lambda_1}{\gamma^2}$$

$$\rho_{\gamma\xi}(2) = \frac{-\lambda_2\beta}{\gamma^2}(\mu_2'\psi_1\Gamma_1 - 2\mu_1'\nu_1)$$

$$\rho_{\beta\xi}(1) = -\Gamma_1\lambda_1$$

$$\rho_{\beta\xi}(2) = \lambda_2(\mu_2'\Gamma_1 - 2\mu_1'\nu_2),$$

and

$$\tilde{\psi}_\alpha = \psi'_\alpha + \psi^2_\alpha, \qquad \nu_1 = \beta\psi_2\Gamma_2 + \xi\psi_1\Gamma_1,$$

$$\lambda_\alpha = \frac{\alpha}{\mu'^2_\alpha}, \qquad \nu_2 = \beta\Gamma_2 + \xi\Gamma_1,$$

with $\psi'_\alpha = \psi'(\alpha/\gamma + 1)$ the trigamma function (Abramowitz and Stegun, 1964, p. 260) indexed to the size-biased order α.

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