Appendix A

Derivatives of the Benefit Functions

This appendix provides the analytical derivation of the first and second order partial derivatives of the benefit functions introduced in Chapter 4. The initial section of this appendix presents derivatives of the downward sloping benefit functions while the second section corresponds to the constant price functions. The expressions presented here are part of a library of functions that are used for the assemblage of the gradient vector and Hessian matrix introduced in Chapter 5, Method of Solution.

Derivatives of the Downward Sloping Benefit Functions

Variable-Head Hydropower Benefit Function

HPW^{VH} - First Partial Derivatives

The benefit function for a variable-head powerplant, equation (4.9), includes three types of decision variables: 1) the powerplant release (*T* variable), 2) controlled releases (*XR* term) other than the powerplant release, and 3) controlled inflows (*XI* term) (figures 3.1 and 4.1). These controlled outflows and inflows are part of equation (4.9a), and therefore subject to differentiation. Three cases of first order partial derivatives (FPD) follow.

Case T FPD with respect to powerplant release *T* for any time step *i*. The role of T_i in the FPD is reflected by the first two lines of (A.1). The release T_i also appears implicitly in (4.9a) as a reservoir outflow for time periods subsequent to *i*, from *i+1* to the end of the optimization horizon *np*. This portion of the derivative is given by the last term in (A.1).

$$\begin{aligned}\frac{\delta B^{HPW^{VH}}}{\delta T_i} = {} & \eta \frac{p_i}{P} \frac{a_2 b_1}{2} \{ [2a_1/b_1 + 2S^o + 2\sum_{k=1}^{i-1}(UI_k + XI_k) - 2\sum_{k=1}^{i-1}(T_k + XR_k + UR_k) \\ & + (UI_i + XI_i - XR_i - UR_i)] \exp(-T_i/b_2) - [T_i \exp(-T_i/b_2)] \} \\ & - \sum_{k=i+1}^{np} \eta \frac{p_k}{P} a_2 b_1 b_2 [1 - \exp(-T_k/b_2)]\end{aligned} \tag{A.1}$$

Case XR FPD with respect to any other controlled release $d_{A_i}, d_{B_i}, ..$ from the same reservoir that supplies water to the powerplant under consideration. For example, the FPD in (A.2) is written with respect to the decision variable d_A. The controlled releases in the FPD at any given time step *i* is reflected by the first term of (A.2). Moreover, the $d_{A_i}, d_{B_i}, ..$ variables also appear as reservoir outflows for time steps *i+1* and forward. This is reflected by the second term.

$$\frac{\delta B^{HPW^{VH}}}{\delta d_{A_i}} = -\eta \frac{p_i}{P} \frac{a_2 b_1 b_2}{2} [1 - \exp(-T_i/b_2)] - \sum_{k=i+1}^{np} \eta \frac{p_k}{P} a_2 b_1 b_2 [1 - \exp(-T_k/b_2)] \quad \text{(A.2)}$$

Case XI FPD with respect to any of the upstream decision variables $d_A^u, d_B^u, ..$ flowing into the reservoir that regulates flows for the powerplant of interest. For example, the first part of (A.3) reflects the role of d_A^u in the FPD at a given time step *i*. Moreover, the variables $d_A^u, d_B^u, ..$ also appear as reservoir inflows in (4.9a) for time *i+1* and forward. This is reflected by the second term.

$$\frac{\delta B^{HPW^{VH}}}{\delta d_{A_i}^u} = \{\eta \frac{p_i}{P} \frac{a_2 b_1 b_2}{2} [1 - \exp(-T_i/b_2)] + \sum_{k=i+1}^{np} \eta \frac{p_k}{P} a_2 b_1 b_2 [1 - \exp(-T_k/b_2)]\} c_A^u \quad \text{(A.3)}$$

where c_A^u is the coefficient of the upstream decision variable d_A^u (controlled inflow) for which the FPD was written.

HPW^{VH} - Second Partial Derivatives

The second order partial derivatives (SPD) presented in this subsection are from equations (A.1), (A.2), and (A.3). In each case we provide a reference to the relative position of the SPD within the global Hessian matrix as discussed in Chapter 6, Mathematical Connectivity of System Components.

Case T-T FPD and SPD with respect to the same variable *T*. Three sub-cases are considered:

1. component submatrix—diagonal terms

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta T_i \, \delta T_i} = -\eta \frac{p_i}{P} \frac{a_2 b_1}{b_2} (a_1/b_1 + b_2/2 + \bar{S}_i) \exp(-T_i/b_2) \quad \text{(A.4)}$$

2. component submatrix—off-diagonal terms (lower-triangular)

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta T_j \, \delta T_i} = -\eta \frac{p_i}{P} a_2 b_1 \exp(-T_i/b_2) \qquad \text{.....} \; for \; j < i \quad \text{(A.5)}$$

3. component submatrix—off-diagonal terms (upper-triangular)

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta T_j \, \delta T_i} = -\eta \frac{p_j}{P} a_2 b_1 \exp(-T_j/b_2) \qquad \text{.....} \; for \; j > i \quad \text{(A.6)}$$

Case T-XR FPD with respect to variable *T*, and SPD with respect to other controlled reservoir release, represented here by d_A. Two sub-cases are considered:

1. link submatrix—diagonal terms

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta d_{A_i}\,\delta T_i} = -\eta \frac{p_i}{P} \frac{a_2 b_1}{2} \exp(-T_i/b_2) \tag{A.7}$$

2. link submatrix—off-diagonal terms (lower-triangular)

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta d_{A_j}\,\delta T_i} = -\eta \frac{p_i}{P} a_2 b_1 \exp(-T_i/b_2) \qquad \ldots\ldots\ for\ j < i \tag{A.8}$$

Case XR-T FPD with respect to other controlled reservoir release d_A, and SPD with respect to *T*. Two sub-cases are considered:

1. link submatrix—diagonal terms

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta T_i\,\delta d_{A_i}} = -\eta \frac{p_i}{P} \frac{a_2 b_1}{2} \exp(-T_i/b_2) \tag{A.9}$$

2. link submatrix—off-diagonal terms (upper-triangular)

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta T_j\,\delta d_{A_i}} = -\eta \frac{p_j}{P} a_2 b_1 \exp(-T_j/b_2) \qquad \ldots\ldots\ for\ j > i \tag{A.10}$$

Case T-XI FPD with respect to *T* and SPD with respect to a controlled reservoir inflow, for example d_A^u. Two sub-cases are considered:

1. link submatrix—diagonal terms

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta d_{A_i}^u\,\delta T_i} = \eta \frac{p_i}{P} \frac{a_2 b_1}{2} \exp(-T_i/b_2)\ c_A^u \tag{A.11}$$

2. link submatrix—off-diagonal terms (lower-triangular)

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta d_{A_j}^u\,\delta T_i} = \eta \frac{p_i}{P} a_2 b_1 \exp(-T_i/b_2)\ c_A^u \qquad \ldots\ldots\ for\ j < i \tag{A.12}$$

where c_A^u is the coefficient of the upstream decision d_A^u.

Case XI-T FPD with respect to a controlled reservoir inflow, for example d_A^u, and SPD with respect to variable T. Two sub-cases are considered:

1. link submatrix—diagonal terms

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta T_i \, \delta d_{A_i}^u} = \eta \frac{p_i}{P} \frac{a_2 b_1}{2} \exp(-T_i/b_2) \; c_A^u \tag{A.13}$$

2. link submatrix—off-diagonal terms (upper-triangular)

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta T_j \, \delta d_{A_i}^u} = \eta \frac{p_j}{P} a_2 b_1 \exp(-T_j/b_2) \; c_A^u \qquad \text{.....} \; for \; j > i \tag{A.14}$$

where c_A^u is the coefficient corresponding to the upstream decision variable d_A^u (controlled inflow) for which the SPD was written.

Fixed-Head Hydropower Benefit Function

HPW^{FH} - First Partial Derivatives

The benefit function for a fixed-head powerplant, equation (4.14), admits only one case of first partial derivative:

Case XI FPD with respect to any of the upstream decision variables contained in the term T_i. For example, writing the derivative for the decision set d_A^u .

$$\frac{\delta B^{HPW^{FH}}}{\delta d_{A_i}^u} = \eta \frac{p_i}{P} a_1 a_2 \exp(-T_i/b_2) \; c_A^u \tag{A.15}$$

where c_A^u is the coefficient of the upstream decision variable d_A^u.

HPW^{FH} - Second Partial Derivatives

The second partial derivatives from equation (A.15) should consider all possible combinations of the controlled inflows. Two sub-cases are contemplated:

Case XI **1**. Component submatrix—diagonal terms. FPD and SPD with respect to the same variable, for instance d_A^u .

$$\frac{\delta^2 B^{HPW^{FH}}}{\delta d_{A_i}^u \, \delta d_{A_i}^u} = -\eta \frac{p_i}{P} \frac{a_1 a_2}{b_2} \exp(-T_i/b_2) \; (c_A^u)^2 \tag{A.16}$$

2. Component submatrix—diagonal terms. FPD and SPD taken with respect to different variables, for instance, d_A^u and d_B^u .

$$\frac{\delta^2 B^{HPW^{FH}}}{\delta d_{B_i}^u \, \delta d_{A_i}^u} = -\eta \frac{p_i}{P} \frac{a_1 a_2}{b_2} \exp(-T_i/b_2) \; c_A^u \, c_B^u \tag{A.17}$$

Irrigation Benefit Function

IRR - First Partial Derivatives

The benefit function for an agricultural demand area, equation (4.19), admits a single case of first partial derivative:

Case XI FPD with respect to any of the upstream decision variables contained in the term A_i. For instance, writing the derivative for the variable d_A^u .

$$\frac{\delta B^{IRR}}{\delta d_{A_i}^u} = a_{3_i} \exp(-A_i/b_{3_i}) \; c_A^u \tag{A.18}$$

where c_A^u is the coefficient corresponding to the upstream decision variable d_A^u that flows into the irrigation zone.

IRR - Second Partial Derivatives

The second partial derivatives are derived from equation (A.18). Two sub-cases are considered:

Case XI **1**. Component submatrix—diagonal terms. FPD and SPD taken with respect to the same variable. In this case, the decision set d_A^u is used.

$$\frac{\delta^2 B^{IRR}}{\delta d_{A_i}^u \, \delta d_{A_i}^u} = -\frac{a_{3_i}}{b_{3_i}} \exp(-A_i/b_{3_i}) \; (c_A^u)^2 \tag{A.19}$$

2. Component submatrix—diagonal terms. FPD and SPD taken with respect to two different control variables. For example, sets d_A^u and d_B^u .

$$\frac{\delta^2 B^{IRR}}{\delta d_{B_i}^u \, \delta d_{A_i}^u} = -\frac{a_{3_i}}{b_{3_i}} \exp(-A_i/b_{3_i}) \; c_A^u \, c_B^u \tag{A.20}$$

Municipal and Industrial Benefit Function

M&I - First Partial Derivatives
The benefit function for a municipal and industrial demand area, equation (4.24), admits a single case of first partial derivative:

Case XI FPD with respect to any of the upstream decision variables contained in the term D_i. For example, taking the derivative with respect to d_A^u we obtain

$$\frac{\delta B^{M\&I}}{\delta d_{A_i}^u} = a_{4_i} \exp(D_i / b_{4_i})\ c_A^u \tag{A.21}$$

where c_A^u is the coefficient of the decision variable of interest.

M&I - Second Partial Derivatives
Second partial derivatives are derived from (A.21). Two cases are contemplated:

Case XI **1**. Component submatrix—diagonal terms. FPD and SPD taken with respect to the same variable, for instance, d_A^u .

$$\frac{\delta^2 B^{M\&I}}{\delta d_{A_i}^u\ \delta d_{A_i}^u} = -\frac{a_{4_i}}{b_{4_i}} \exp(-D_i / b_{4_i})\ (c_A^u)^2 \tag{A.22}$$

2. Component submatrix—diagonal terms. FPD and SPD taken with respect to two different control variables, for example, d_A^u and d_B^u .

$$\frac{\delta^2 B^{M\&I}}{\delta d_{B_i}^u\ \delta d_{A_i}^u} = -\frac{a_{4_i}}{b_{4_i}} \exp(D_i / b_{4_i})\ c_A^u\ c_B^u \tag{A.23}$$

where now c_A^u and c_B^u are the coefficients corresponding to the two decision variables of interest.

Instream Water Recreation Benefit Function

IRA - First Partial Derivatives
The benefit function for an instream water recreation activity, equation (4.29), admits a single case of first partial derivative:

Case XI FPD with respect to any of the upstream decision variables contained in the term R_i. For example, writing the derivative with respect to d_A^u yields

$$\frac{\delta B^{IRA}}{\delta d_{A_i}^u} = (a_{5_i} - b_{5_i} R_i)\, c_A^u \tag{A.24}$$

where c_A^u is the coefficient corresponding to the decision variable d_A^u.

IRA - Second Partial Derivatives

The second partial derivatives are derived from equation (A.24). Two cases are contemplated:

Case XI **1**. Component submatrix—diagonal terms. FPD and SPD taken with respect to the same variable. In this example d_A is used.

$$\frac{\delta^2 B^{IRA}}{\delta d_{A_i}^u\, \delta d_{A_i}^u} = -\, b_{5_i} (c_A^u)^2 \tag{A.25}$$

2. Component submatrix—diagonal terms. FPD and SPD taken with respect to two different control variables, for example, variables d_A^u and d_B^u.

$$\frac{\delta^2 B^{IRA}}{\delta d_{B_i}^u\, \delta d_{A_i}^u} = -\, b_{5_i}\, c_A^u\, c_B^u \tag{A.26}$$

where c_A^u and c_B^u correspond to the decision variables of interest.

Reservoir Recreation Benefit Function

The benefit function for reservoir recreation (RRA), equation (4.30), admits partial derivatives with respect to all controlled releases and inflows to the reservoir where recreation takes place. The partial derivatives presented herein are organized in a form similar to hydropower.

RRA - First Partial Derivatives

Case XR FPD of the reservoir recreation benefit function with respect to any controlled release $d_A, d_B, ..$ For example, the FPD in (A.27) is written with respect to the decision variable d_A. The controlled release during time step i yields the first term of (A.27). Moreover, d_{A_i} also appears as a reservoir outflow for time steps $i+1$ and forward. This is expressed by the second term of (A.27).

$$\frac{\delta B^{RRA}}{\delta d_{A_i}} = -\frac{1}{2}\frac{a_{6_i} b_{6_i}}{\cosh^2(b_{6_i}\bar{S}_i - c_{6_i})} - \sum_{k=i+1}^{np} \frac{a_{6_k} b_{6_k}}{\cosh^2(b_{6_k}\bar{S}_k + c_{6_k})} \tag{A.27}$$

Case XI FPD of the reservoir recreation benefit function with respect to any controlled reservoir inflow $d_A^u, d_B^u, ..$ For example, the FPD in (A.28) is written with respect to the decision variable d_A^u . The first term of (A.28) reflects the role of $d_{A_i}^u$ as a reservoir inflow during time step *i*. Moreover, the $d_{A_i}^u$ also appears as a reservoir inflow for time steps *i+1* and forward. This is reflected by the second term of (A.28),

$$\frac{\delta B^{RRA}}{\delta d_{A_i}^u} = \left(\frac{1}{2}\frac{a_{6_i} b_{6_i}}{\cosh^2(b_{6_i}\bar{S}_i - c_{6_i})} + \sum_{k=i+1}^{np} \frac{a_{6_k} b_{6_k}}{\cosh^2(b_{6_k}\bar{S}_k + c_{6_k})} \right) c_A^u \tag{A.28}$$

where c_A^u is the coefficient of the upstream decision variable d_A^u (controlled inflow) for which the FPD was written.

RRA - Second Partial Derivatives

Case XR-XR Indicates the case where the FPD and SPD are taken both with respect to controlled releases. The following SPDs are derived from equation A.27. Three subcases are considered according to the time steps under consideration. In each case we provide a reference to the relative position of the SPD within the global Hessian matrix.

1. component submatrix—diagonal terms. FPD and SPD taken at the same time step *i*

$$\frac{\delta^2 B^{RRA}}{\delta d_{B_i} \delta d_{A_i}} = -\frac{1}{2} a_{6_i} b_{6_i}^2 \frac{\tanh(b_{6_i}\bar{S}_i - c_{6_i})}{\cosh^2(b_{6_i}\bar{S}_i - c_{6_i})} - 2\sum_{k=i+1}^{np} a_{6_k} b_{6_k}^2 \frac{\tanh(b_{6_k}\bar{S}_k - c_{6_k})}{\cosh^2(b_{6_k}\bar{S}_k - c_{6_k})} \tag{A.29}$$

2. component submatrix—off-diagonal terms (lower-triangular). SPD taken at time steps earlier than the FPD, i.e., for... $j < i$

$$\frac{\delta^2 B^{RRA}}{\delta d_{B_j} \delta d_{A_i}} = -a_{6_i} b_{6_i}^2 \frac{\tanh(b_{6_i}\bar{S}_i - c_{6_i})}{\cosh^2(b_{6_i}\bar{S}_i - c_{6_i})} - 2\sum_{k=i+1}^{np} a_{6_k} b_{6_k}^2 \frac{\tanh(b_{6_k}\bar{S}_k - c_{6_k})}{\cosh^2(b_{6_k}\bar{S}_k - c_{6_k})} \tag{A.30}$$

3. component submatrix—off-diagonal terms (upper-triangular). SPD taken at time steps later than the FPD, i.e., for... $j > i$

$$\frac{\delta^2 B^{RRA}}{\delta d_{B_j} \delta d_{A_i}} = -a_{6_j} b_{6_j}^2 \frac{\tanh(b_{6_j}\bar{S}_j - c_{6_j})}{\cosh^2(b_{6_j}\bar{S}_j - c_{6_j})} - 2 \sum_{k=j+1}^{np} a_{6_k} b_{6_k}^2 \frac{\tanh(b_{6_k}\bar{S}_k - c_{6_k})}{\cosh^2(b_{6_k}\bar{S}_k - c_{6_k})} \tag{A.31}$$

Note: equations (A.29) through (A.31) also apply when the SPD is taken with respect to the same controlled release as the FPD (e.g., $\delta^2 B^{RRA} / \delta d_A \delta d_A$)

Case XI-XI Indicates the case where the FPD and SPD are taken both with respect to controlled inflows. The following SPDs are derived from equation (A.28). Three subcases are considered according to the time steps under consideration. In each case we provide a reference to the relative position of the SPD within the global Hessian matrix.

1. component submatrix—diagonal terms. FPD and SPD taken at the same time step i

$$\frac{\delta^2 B^{RRA}}{\delta d_{B_i}^u \delta d_{A_i}^u} = \left(-\frac{1}{2} a_{6_i} b_{6_i}^2 \frac{\tanh(b_{6_i}\bar{S}_i - c_{6_i})}{\cosh^2(b_{6_i}\bar{S}_i - c_{6_i})} - 2 \sum_{k=i+1}^{np} a_{6_k} b_{6_k}^2 \frac{\tanh(b_{6_k}\bar{S}_k - c_{6_k})}{\cosh^2(b_{6_k}\bar{S}_k - c_{6_k})} \right) c_A^u c_B^u \tag{A.32}$$

2. component submatrix—off-diagonal terms (lower-triangular). SPD taken at time steps earlier than the FPD, i.e., for... $j < i$

$$\frac{\delta^2 B^{RRA}}{\delta d_{B_j}^u \delta d_{A_i}^u} = \left(-a_{6_i} b_{6_i}^2 \frac{\tanh(b_{6_i}\bar{S}_i - c_{6_i})}{\cosh^2(b_{6_i}\bar{S}_i - c_{6_i})} - 2 \sum_{k=i+1}^{np} a_{6_k} b_{6_k}^2 \frac{\tanh(b_{6_k}\bar{S}_k - c_{6_k})}{\cosh^2(b_{6_k}\bar{S}_k - c_{6_k})} \right) c_A^u c_B^u \tag{A.33}$$

3. component submatrix—off-diagonal terms (upper-triangular). SPD taken at time steps later than the FPD, i.e., for... $j > i$

$$\frac{\delta^2 B^{RRA}}{\delta d_{B_j}^u \delta d_{A_i}^u} = \left(-a_{6_j} b_{6_j}^2 \frac{\tanh(b_{6_j}\bar{S}_j - c_{6_j})}{\cosh^2(b_{6_j}\bar{S}_j - c_{6_j})} - 2 \sum_{k=i+1}^{np} a_{6_k} b_{6_k}^2 \frac{\tanh(b_{6_k}\bar{S}_k - c_{6_k})}{\cosh^2(b_{6_k}\bar{S}_k - c_{6_k})} \right) c_A^u c_B^u \tag{A.34}$$

Note: equations (A.32) to (A.34) also apply when the SPD is taken with respect to the same controlled inflow as the FPD (e.g., $\delta^2 B^{RRA} / \delta d_A^u \delta d_A^u$)

Case XR-XI Indicates the case where the FPD and SPD are taken with respect to a controlled release and a controlled inflow, respectively. The following SPDs are derived from equation (A.27). In each case we provide a reference to the relative position of the SPD within the global Hessian matrix. Three subcases exist according to the time steps under consideration:

1. component submatrix—diagonal terms. FPD and SPD taken at the same time step *i*

$$\frac{\delta^2 B^{RRA}}{\delta d_{B_i}^u \, \delta d_{A_i}} = \left(\frac{1}{2} a_{6_i} b_{6_i}^2 \frac{\tanh(b_{6_i}\bar{S}_i - c_{6_i})}{\cosh^2(b_{6_i}\bar{S}_i - c_{6_i})} + 2 \sum_{k=i+1}^{np} a_{6_k} b_{6_k}^2 \frac{\tanh(b_{6_k}\bar{S}_k - c_{6_k})}{\cosh^2(b_{6_k}\bar{S}_k - c_{6_k})} \right) c_B^u \quad \text{(A.35)}$$

2. component submatrix—off-diagonal terms (lower-triangular). SPD taken at time steps earlier than the FPD, i.e., for... $j < i$

$$\frac{\delta^2 B^{RRA}}{\delta d_{B_j}^u \, \delta d_{A_i}} = \left(a_{6_i} b_{6_i}^2 \frac{\tanh(b_{6_i}\bar{S}_i - c_{6_i})}{\cosh^2(b_{6_i}\bar{S}_i - c_{6_i})} + 2 \sum_{k=i+1}^{np} a_{6_k} b_{6_k}^2 \frac{\tanh(b_{6_k}\bar{S}_k - c_{6_k})}{\cosh^2(b_{6_k}\bar{S}_k - c_{6_k})} \right) c_B^u \quad \text{(A.36)}$$

3. component submatrix—off-diagonal terms (upper-triangular). SPD taken at time steps later than the FPD, i.e., for... $j > i$

$$\frac{\delta^2 B^{RRA}}{\delta d_{B_j}^u \, \delta d_{A_i}} = \left(a_{6_j} b_{6_j}^2 \frac{\tanh(b_{6_j}\bar{S}_j - c_{6_j})}{\cosh^2(b_{6_j}\bar{S}_j - c_{6_j})} + 2 \sum_{k=j+1}^{np} a_{6_k} b_{6_k}^2 \frac{\tanh(b_{6_k}\bar{S}_k - c_{6_k})}{\cosh^2(b_{6_k}\bar{S}_k - c_{6_k})} \right) c_B^u \quad \text{(A.37)}$$

Case XI-XR Indicates the case when the FPD and SPD are taken with respect to a controlled inflow and a controlled release, respectively. The following SPDs are derived from equation (A.28). In each case we provide a reference to the relative position of the SPD within the global Hessian matrix. Three subcases exist according to the time steps under consideration:

1. component submatrix—diagonal terms. FPD and SPD taken at the same time step *i*

$$\frac{\delta^2 B^{RRA}}{\delta d_{B_i} \, \delta d_{A_i}^u} = \left(\frac{1}{2} a_{6_i} b_{6_i}^2 \frac{\tanh(b_{6_i}\bar{S}_i - c_{6_i})}{\cosh^2(b_{6_i}\bar{S}_i - c_{6_i})} + 2 \sum_{k=i+1}^{np} a_{6_k} b_{6_k}^2 \frac{\tanh(b_{6_k}\bar{S}_k - c_{6_k})}{\cosh^2(b_{6_k}\bar{S}_k - c_{6_k})} \right) c_A^u \quad \text{(A.38)}$$

2. component submatrix—off-diagonal terms (lower-triangular). SPD taken at time steps earlier than the FPD, i.e., for... $j < i$

$$\frac{\delta^2 B^{RRA}}{\delta d_{B_j}\,\delta d^u_{A_i}} = \left(a_{6_i} b_{6_i}^2 \frac{\tanh(b_{6_i}\bar{S}_i - c_{6_i})}{\cosh^2(b_{6_i}\bar{S}_i - c_{6_i})} + 2 \sum_{k=i+1}^{np} a_{6_k} b_{6_k}^2 \frac{\tanh(b_{6_k}\bar{S}_k - c_{6_k})}{\cosh^2(b_{6_k}\bar{S}_k - c_{6_k})} \right) c_A^u \qquad \text{(A.39)}$$

3. component submatrix—off-diagonal terms (upper-triangular). SPD taken at time steps later than the FPD, i.e., for... $j > i$

$$\frac{\delta^2 B^{RRA}}{\delta d_{B_j}\,\delta d^u_{A_i}} = \left(a_{6_j} b_{6_j}^2 \frac{\tanh(b_{6_j}\bar{S}_j - c_{6_j})}{\cosh^2(b_{6_j}\bar{S}_j - c_{6_j})} + 2 \sum_{k=j+1}^{np} a_{6_k} b_{6_k}^2 \frac{\tanh(b_{6_k}\bar{S}_k - c_{6_k})}{\cosh^2(b_{6_k}\bar{S}_k - c_{6_k})} \right) c_A^u \qquad \text{(A.40)}$$

Derivatives of the Constant Price Benefit Functions

Variable-Head Hydropower Benefit Function

HPW^{VH} - First Partial Derivatives

Case T FPD with respect to powerplant release T for any time step i.

$$\begin{aligned}\frac{\delta B^{HPW^{VH}}}{\delta T_i} = \eta \frac{p_i}{P} \frac{a_2 b_1}{2} \{ & \frac{2a_1}{b_1} + 2S^o + 2\sum_{k=1}^{i-1}(UI_k + XI_k) - 2\sum_{k=1}^{i-1}(T_k + XR_k + UR_k) \\ & + (UI_i + XI_i - XR_i - UR_i - T_i)\} - \sum_{k=i+1}^{np} \eta \frac{p_k}{P} a_2 b_1 T_k\end{aligned} \qquad \text{(A.41)}$$

Case XR FPD with respect to any other controlled release $d_{A_i}, d_{B_i}, ..$ from the same reservoir that supplies water to the powerplant under consideration.

$$\frac{\delta B^{HPW^{VH}}}{\delta d_{A_i}} = -\eta \frac{p_i}{P} \frac{a_2 b_1}{2} T_i - \sum_{k=i+1}^{np} \eta \frac{p_k}{P} a_2 b_1 T_k \qquad \text{(A.42)}$$

Case XI FPD with respect to any of the upstream decision variables $d^u_A, d^u_B, ..$ flowing into the reservoir that regulates flows for the powerplant of interest.

$$\frac{\delta B^{HPW^{VH}}}{\delta d_{A_i}^{u}} = \{\eta \frac{p_i}{P} \frac{a_2 b_1}{2} T_i + \sum_{k=i+1}^{np} \eta \frac{p_k}{P} a_2 b_1 T_k\} c_A^u \tag{A.43}$$

HPW^{VH} - Second Partial Derivatives

Case T-T FPD and SPD with respect to the same variable *T*. Three sub-cases are considered:

1. component sub-matrix—diagonal terms.

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta T_i \, \delta T_i} = -\eta \frac{p_i}{P} \frac{a_2 b_1}{2} \tag{A.44}$$

2. component sub-matrix—off-diagonal terms (lower-triangular)

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta T_j \, \delta T_i} = -\eta \frac{p_i}{P} a_2 b_1 \qquad \text{.....} \; for \; j < i \tag{A.45}$$

3. component sub-matrix—off-diagonal terms (upper-triangular)

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta T_j \, \delta T_i} = -\eta \frac{p_j}{P} a_2 b_1 \qquad \text{.....} \; for \; j > i \tag{A.46}$$

Case T-XR FPD with respect to variable *T* and SPD with respect to other controlled reservoir release, represented here by d_A. Two sub-cases are considered:

1. link sub-matrix—diagonal terms.

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta d_{A_i} \, \delta T_i} = -\eta \frac{p_i}{P} \frac{a_2 b_1}{2} \tag{A.47}$$

2. link sub-matrix—off-diagonal terms (lower-triangular)

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta d_{A_j} \, \delta T_i} = -\eta \frac{p_i}{P} a_2 b_1 \qquad \text{.....} \; for \; j < i \tag{A.48}$$

Case XR-T FPD with respect to other controlled reservoir release d_A and SPD with respect to *T*. Two sub-cases are considered:

1. link sub-matrix—diagonal terms

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta T_i \, \delta d_{A_i}} = -\eta \frac{p_i}{P} \frac{a_2 b_1}{2} \tag{A.49}$$

2. link sub-matrix—off-diagonal terms (upper-triangular)

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta T_j \, \delta d_{A_i}} = -\eta \frac{p_j}{P} a_2 b_1 \qquad \text{.....} \; for \; j > i \tag{A.50}$$

Case T-XI FPD with respect to T and SPD with respect to a controlled reservoir inflow, for example d_A^u. Two sub-cases are considered:

1. link sub-matrix—diagonal terms

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta d_{A_i}^u \, \delta T_i} = \eta \frac{p_i}{P} \frac{a_2 b_1}{2} \; c_A^u \tag{A.51}$$

2. link sub-matrix—off-diagonal terms (lower-triangular)

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta d_{A_j}^u \, T_i} = \eta \frac{p_i}{P} a_2 b_1 \; c_A^u \qquad \text{.....} \; for \; j < i \tag{A.52}$$

Case XI-T FPD with respect to a controlled reservoir inflow, for example d_A^u, and SPD with respect to variable T. Two sub-cases are considered:

1. link sub-matrix—diagonal terms.

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta T_i \, \delta d_{A_i}^u} = \eta \frac{p_i}{P} \frac{a_2 b_1}{2} \; c_A^u \tag{A.53}$$

2. link sub-matrix—off-diagonal terms (upper-triangular)

$$\frac{\delta^2 B^{HPW^{VH}}}{\delta T_j \, \delta d_{A_i}^u} = \eta \frac{p_j}{P} a_2 b_1 \; c_A^u \qquad \text{.....} \; for \; j > i \tag{A.54}$$

Fixed-Head Hydropower Benefit Function

HPW^{FH} - First Partial Derivatives

Case XI FPD with respect to any upstream decision variable contained in the term T_i.

$$\frac{\delta B^{HPW^{FH}}}{\delta d_{A_i}^u} = \eta \frac{p_i}{P} a_1 a_2 \exp(-T_i / b_2) \; c_A^u \tag{A.55}$$

HPW^{FH} - Second Partial Derivatives

SPDs are equal to zero.

Irrigation Benefit Function

IRR - First Partial Derivatives

Case XI FPD with respect to any upstream decision variable contained in the term A_i.

$$\frac{\delta B^{IRR}}{\delta d_{A_i}^{u}} = a_{3_i} \, c_A^u \tag{A.56}$$

IRR - Second Partial Derivatives

SPDs are equal to zero.

Municipal and Industrial Benefit Function

M&I - First Partial Derivatives

Case XI FPD with respect to any upstream decision variable contained in the term D_i.

$$\frac{\delta B^{M\&I}}{\delta d_{A_i}^{u}} = a_{4_i} \exp(D_i / b_{4_i}) \, c_A^u \tag{A.57}$$

M&I - Second Partial Derivatives

SPDs are equal to zero.

Instream Water Recreation Benefit Function

IRA - First Partial Derivatives

Case XI FPD with respect to any upstream decision variable contained in the term R_i.

$$\frac{\delta B^{IRA}}{\delta d_{A_i}^{u}} = a_{5_i} \, c_A^u \tag{A.58}$$

IRA - Second Partial Derivatives

SPDs are equal to zero.

Appendix B

Fitting Exponential Demand Functions

This appendix presents the equations used by AQUARIUS to fit exponential inverse demand functions based on economic information provided by the user. Further discussion regarding this topic is in Chapter 2, Economic Demand Functions, General Concepts.

Cases Considered

Three cases are considered for the analytical fitting of demand curves, described below with the assistance of figure B.1:

Case I: prices and quantities are known at any two points along the demand curve. In general, the two points may represent: (1) the price at which quantity demanded falls to zero, $\{Q_1,P_1\}$ and (2) the quantity at which price equals the existing price, $\{Q_2,P_2\}$.

Case II: price and quantity are known at a single point, $\{Q_2,P_2\}$, in addition to the elasticity at that same point yielding the dimensionless slope ϵ_2.

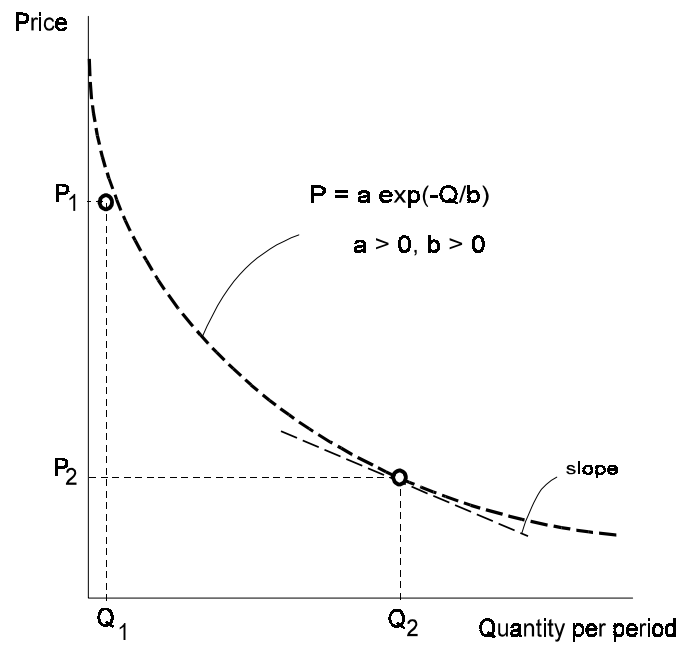

Figure B.1 Information defining an exponential demand.

Case III: is the combination of Cases I and II, in which three pieces of information are specified by the user: the two price-quantity points, $\{Q_1,P_1\}$ and $\{Q_2,P_2\}$, and the elasticity ϵ_2.

Basic Equations

We start the analysis by expressing the exponential inverse demand function *P=a exp(-Q/b)* at the point $\{Q_1,P_1\}$, where demand falls near zero,

$$P_1 = a \exp(-Q_1/b) \tag{B.1}$$

Similarly, the exponential demand function can be written at the equilibrium point $\{Q_2,P_2\}$,

$$P_2 = a \exp(-Q_2/b) \tag{B.2}$$

By definition, the elasticity ϵ of the demand curve is given by Eq.(B.3) (Chapter 2, Economic Demand Functions, General Concepts). In particular, for an exponential demand function, elasticity is expressed by the ratio of the coefficient *b* and the quantity demanded *Q*,

$$\epsilon = \frac{\partial Q/Q}{\partial P/P} = -b/Q \tag{B.3}$$

Computing the Coefficients "*a*" and "*b*"

Each case is analyzed separately:

Case I Given the two paired values $\{Q_1,P_1\}$ and $\{Q_2,P_2\}$, the parameters *a* and *b* can be defined with no error since there is only one model of the form $P = a\ exp(-Q/b)$ that will assume the exact values $f(Q_1)$ and $f(Q_2)$. Working with equations (B.1) into (B.2) yields:

$$a = \frac{P_1}{\exp\left(\dfrac{\ln(P_1/P_2)}{1 - Q_2/Q_1}\right)} \quad ..... \quad for \ \ Q_1 > 0 \tag{B.4}$$

$$b = \frac{Q_2 - Q_1}{\ln(P_1/P_2)} \quad ..... \quad for \ \ Q_1 > 0 \tag{B.5}$$

If one of the points corresponds to the intercept $\{Q_1=0, P=P_1\}$, the coefficients *a* and *b* are given by:

$$a = P_1 \quad ..... \quad for \ \ Q_1 = 0 \tag{B.6}$$

$$b = \frac{Q_2}{\ln(P_1/P_2)} \quad ..... \quad for \ \ Q_1 = 0 \tag{B.7}$$

Case II When one point of the curve is known $\{P_2, Q_2\}$ along with its elasticity ϵ_2, there is a unique solution to the problem. In this case, the coefficients *a* and *b* are expressed by:

$$b = -\epsilon_2 Q_2 \tag{B.8}$$

$$a = \frac{P_2}{\exp(1/\epsilon_2)} \tag{B.9}$$

Case III When two points and the elasticity ϵ at one of those points are known, the fitting problem becomes one of interpolating fitting since there are three pieces of information and only two parameters to estimate. The problem is reduced to one of finding the best estimates of the coefficients *a* and *b* of the exponential function *P= a exp(-Q/b)* using the *Least-Squares* (LS) principle. This involves minimizing the sum of all squares of deviations of the observed points and elasticity from the fitted function.

When the economic information available, price-quantity points and elasticity, are thought to be of unequal reliability, the LS criterion can be modified to require that the squared error terms be multiplied by nonnegative weight factors w_i before the aggregated square error is calculated (*weighted least-squares*). That is, from (B.1), (B.2), and (B.3), the weighted sum of the square of the residuals assumes the form:

$$\begin{aligned} Min \quad S = {} & w_1 [P_1 - a \exp(-Q_1/b)]^2 + w_2 [P_2 - a \exp(-Q_2/b)]^2 \\ & + w_\epsilon [\epsilon_2 - b/Q_2)]^2 \end{aligned} \tag{B.10}$$

where the weighting coefficients w_1, w_2, and w_ϵ, for the two data points and elasticity, respectively, are a measure of the degree of precision or degree of importance of the three pieces of information in determining the coefficients of the demand function. Equation (B.10) is a nonlinear expression that can be minimized using different techniques.

Fitting Example

Figure B.2 shows the dialog-box presented to the user, which is accessed from the Tools menu, to estimate the coefficients of the exponential demand function for the three cases described above.

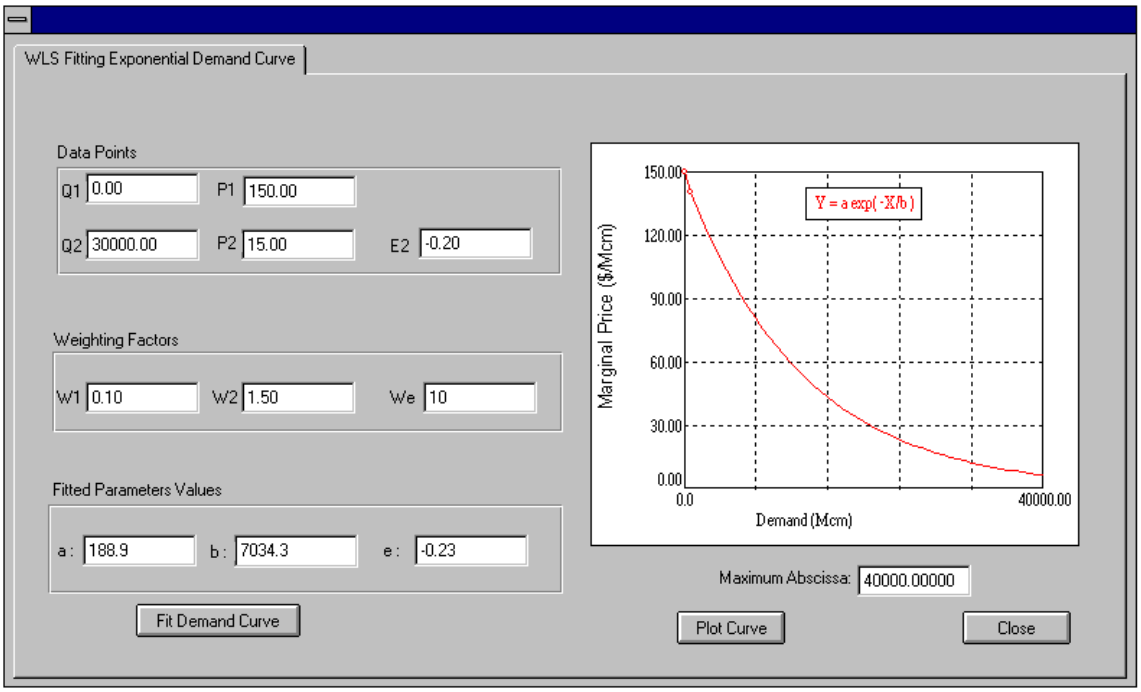

Figure B.2 Graphical user interface (GUI) for fitting an exponential demand curve.

The Data Points window includes the user specified price-quantity data for two points: $\{Q_1, P_1\}$ = {0, 150} and $\{Q_2,P_2\}$ = {30000, 15}; and an estimated regional elasticity ϵ_2 = -0.20. When the set of weights w_1, w_2, and w_3 are entered as {1, 1, 0} (with zero weight for the regional elasticity), the solution is trivial (Case I) yielding a = 150, b = 13029, and the resulting elasticity at Q_2 is ϵ_2 = -0.434. A second run assuming equal weights {1, 1, 1} for the three pieces of information yields the parameters a = 150.02, b = 12959.7, and elasticity at Q_2 equal to -0.43.

Knowing the regional elasticity to be -0.2, the weighting coefficients are changed again to redirect the fitting algorithm to increase the compliance to the regional elasticity value. A new set of weights {0.1, 1.5, 10} forces the LS fitting procedure to yield a demand function with parameters a = 188.9, b = 7034.3, and a computed elasticity ϵ_2 = -0.23 (figure B.2). Note that this value approximates the regional elasticity at the expense of the quality of fitting of the two data points.