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"MISSING"
A Computer Program for the Maximum Likelihood Estimates of the Parameters of the Multivariate Linear Model with Incomplete Measurements

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Abstract

This report describes a computer program that obtains maximum likelihood estimates of the parameters of a multivariate linear model in which all variates may not be measured on each experimental unit. The variates can be: (1) repeated measurements on the same characteristic, (2) different characteristics, or (3) a mixture of repeated measurements and different characteristics.

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INTRODUCTION

THIS TECHNICAL REPORT describes a computer program that obtains maximum likelihood estimates of the parameters of a multivariate linear model with correlated errors when some of the dependent variates are not measured on some of the experimental units.

Our procedure was developed to model the forest growth and yield functions when there are more than two repeated measurements per plot, but not all measurements are made on every plot. The program was developed to obtain solutions to this model, but can also be used to solve several well-known statistical models including the multivariate linear regression model (Anderson 1958), and Zellner's (1962) seemingly unrelated regressions model.

DEFINITIONS AND NOTATIONS USED IN INCOMPLETE MULTIVARIATE MODELS

Suppose we have a study with N plots on which T variates $Y_1 \ldots, Y_t, \ldots Y_T$ can be measured, but not all variates are measured on each plot. The missing measurements may be missing by design or by chance. The N plots can be divided into K disjoint groups $G_1 \ldots, G_k, \ldots G_K$. The k^{th} group consists of the N_k plots on which a subset of T_k of the variates are measured. The variates can be: (1) repeated measurements on the same characteristic, (2) different characteristics, or (3) a mixture of repeated measurements and different characteristics.

As an example, suppose that three variates Y_1, Y_2, and Y_3 are measured. The seven groups that are possible are

Group	G_1	G_2	G_3	G_4	G_5	G_6	G_7
Variates	1	2	3	1,2	1,3	2,3	1,2,3

Groups 1 through 6 are incompletely measured; Group 7 has a complete set of measurements.

Let y_{kti} be the observed value of the variate Y_t on the i^{th} plot in the k^{th} group. The $N_k \times 1$ vector of observations on Y_t in the k^{th} group is defined by:

$$y_{kt} = (y_{kt1} \ldots y_{ktN_k})'.$$

The $N_k T_k \times 1$ composite vector of observed values of variates measured in the k^{th} group is defined by $y_k = (\ldots y'_{kt} \ldots)'$. To illustrate, the overall vector of observations for group 5 is $y_5 = (y'_{51}, y'_{53})'$.

The model for the incomplete composite response vector is written in terms of group vectors and group design matrices as

$$\begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_k \\ \cdot \\ \cdot \\ \cdot \\ y_K \end{bmatrix} = \begin{bmatrix} X_1 \\ \cdot \\ \cdot \\ \cdot \\ X_k \\ \cdot \\ \cdot \\ \cdot \\ X_K \end{bmatrix} \beta + \begin{bmatrix} \epsilon_1 \\ \cdot \\ \cdot \\ \cdot \\ \epsilon_k \\ \cdot \\ \cdot \\ \cdot \\ \epsilon_K \end{bmatrix}$$

where

$$y'_k = (y'_{k1}, \ldots, y'_{kt}, \ldots, y'_{kT})$$

$$X'_k = (X'_{k1}, \ldots, X'_{kt}, \ldots, X'_{kT})$$

and

$$y'_{kt} = [y_{kt1}, \ldots, y_{ktN_k}]$$

$$X'_{kt} = [x'_{kt1}, \ldots, x'_{ktN_k}]$$

$$x_{kti} = [x_{kti1}, \ldots, x_{ktiq}].$$

β is a composite vector of model parameters of length q and ϵ_k is the $T_k N_k \times 1$ vector of random errors associated with y_k.

We assume that the observations on the T_k variates on each plot of group k have a common covariance matrix V_k, while the observations on the N_k plots are mutually independent. Under these assumptions, the covariance matrix for y_k can be written $V_k \otimes I_k$ where $\otimes$ denotes the Kronecker product and I_k is an identity matrix of order N_k.

It is also assumed that response vectors from different groups (y_k, $y_{k'}$, $k \neq k'$) are mutually independent. Therefore, the covariance matrix of the composite response vector $y = (y'_1, \ldots, y'_K)'$ is

$$\Psi = \begin{bmatrix} (V_1 \otimes I_1) & \ldots & 0 \\ \cdot & \ldots & \cdot \\ 0 & \ldots & (V_K \otimes I_K) \end{bmatrix}$$

V_k is a function of the population covariance V, which is the $T \times T$ covariance matrix that would exist among the complete set of T variates.

The form of the design matrix $X = (X'_1, \ldots X'_k, \ldots X'_K)'$ and the vector of parameters, β, depends on the specific model and the variates measured. To illustrate the notation, we show the observational structure for three specific models assuming the three-variate situation introduced previously.

For Sullivan and Clutter's (1972) yield model, the variates are repeated measurements on a plot. The observational equations for the complete group with three measurements are

$$\begin{bmatrix} y_{71} \\ y_{72} \\ y_{73} \end{bmatrix} = \begin{bmatrix} X_{71} \\ X_{72} \\ X_{73} \end{bmatrix} \beta$$

whereas the observational structure for group 5, where only variates 1 and 3 are measured, is

$$\begin{bmatrix} y_{51} \\ \\ y_{53} \end{bmatrix} = \begin{bmatrix} X_{51} \\ \\ X_{53} \end{bmatrix} \beta$$

If N_k plots are measured at time t, y_{kt} is a column vector of length N_k, and X_{kt} is $N_{kt} \times q$. The $q \times 1$ vector of parameters, β, is the same for all measurement periods. The observed values of the independent variables in X_{kt} change over time.

The observational equations for the complete group in a multivariate linear regression model can be written

$$\begin{bmatrix} y_{71} \\ y_{72} \\ y_{73} \end{bmatrix} = \begin{bmatrix} X_{71} \\ X_{72} \\ X_{73} \end{bmatrix} \beta = \begin{bmatrix} X^*_7 & 0 & 0 \\ 0 & X^*_7 & 0 \\ 0 & 0 & X^*_7 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

The observational equations for group 5 are

$$\begin{bmatrix} y_{51} \\ y_{53} \end{bmatrix} = \begin{bmatrix} X_{51} \\ X_{53} \end{bmatrix} \beta = \begin{bmatrix} X^*_5 & 0 & 0 \\ 0 & 0 & X^*_5 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

Notice that: (1) the design matrix X_{kt} is an $N_k \times q$ matrix composed of the $N_k \times q_t$ submatrix X^*_k and null matrices 0, and that (2) $q_t = q_{t'}$. The matrix X_{kt} differs from $X_{kt'}$ by the position of the submatrix X^*_k. The vector of parameters, β_t, differs for each variate.

In the third model, both the design matrices, X_{kt}, and the vectors of parameters, β_t, are different for each variate. The observational equations for the complete group of seemingly unrelated regressions (Zellner 1962) are

$$\begin{bmatrix} y_{71} \\ y_{72} \\ y_{73} \end{bmatrix} = \begin{bmatrix} X^*_{71} & 0 & 0 \\ 0 & X^*_{72} & 0 \\ 0 & 0 & X^*_{73} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

The equations for the incomplete group are

$$\begin{bmatrix} y_{51} \\ y_{53} \end{bmatrix} = \begin{bmatrix} X^*_{51} & 0 & 0 \\ 0 & 0 & X^*_{53} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

The covariance matrix for a group depends on the variates measured in that group. For example, since group 7 is complete, the covariance matrix of the observations on an experimental unit is V. Group 5 has measurements on variates 1 and 3 only. Therefore, the covariance matrix for observations on an experimental unit in group 5 is

$$V_5 = \begin{bmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{bmatrix}$$

Because of the assumed dependence of the measurements, ordinary least squares (OLS) would not yield efficient estimates of β. We assume that the vector y has a multivariate normal distribution with $\mathcal{E}(y) = X\beta$ and covariance matrix Ψ, and find values of β and V that simultaneously maximize the likelihood function

$$L = (2\pi)^{-\sum_{k=1}^{K} N_k T_k/2} |\Psi|^{-1/2} e^{-1/2 Q}$$

where

$$Q = \sum_{k=1}^{K} (y_k - X_k \beta)' \Psi^{-1}_k (y_k - X_k \beta)$$

and $\Psi_k = V_k \otimes I_k$.

The symbol $|\Psi|$ denotes the determinant of the matrix Ψ.

The logarithm of the likelihood function is:

$$\ln L = -\tfrac{1}{2}\ln(2\pi)\sum_{k=1}^{K} N_k T_k$$

$$+ \tfrac{1}{2}\sum_{k=1}^{K} N_k \ln V^{-1}_k - \tfrac{1}{2}Q.$$

The likelihood equations are the derivatives of lnL taken with respect to β and the elements of V. The maximum likelihood estimators of β and V are those values for which the likelihood equations are simultaneously equal to zero. The expressions for the likelihood equations will be included in a forthcoming publication by Seegrist and Arner.

CALCULATING THE MAXIMUM LIKELIHOOD ESTIMATES

The likelihood equations are nonlinear, and iterative procedures must be used to obtain a solution. We begin with the OLS estimator $\hat{\beta}_o = (X'X)^{-1} X'Y$, and then calculate the sample variances and covariances for each group:

$$\hat{\sigma}_{ktuo} = (y_{kt} - X_{kt}\hat{\beta}_o)'(y_{ku} - X_{ku}\hat{\beta}_o)/N_k.$$

The initial estimates for the elements of V that we use are the average variances and covariances

$$\hat{\sigma}_{tuo} = \Sigma N_k \hat{\sigma}_{ktuo}/\Sigma N_k.$$

These estimates of the variances and covariances are stored in a $[T(T+1)/2 \times 1]$ vector denoted by σ. Denote the overall vector of parameters by $\theta = (\beta', \sigma')'$.

We have several procedures for estimating the vector of regression coefficients and the variances and covariances. The estimate of θ at iteration i is $\hat{\theta}_i = \hat{\theta}_{i-1} - Q^{-1}{}_{i-1} S_{i-1}$. S_{i-1} denotes the vector of first derivatives of lnL — with respect to β and with respect to the elements of Σ — evaluated at $\hat{\theta}_{i-1}$. Q_{i-1} depends on which procedure is used. For the Newton-Raphson procedure, Q_{i-1} is the matrix of second derivatives evaluated at $\hat{\theta}_{i-1}$. For the Fisher scoring procedure, Q_{i-1} is the expected value of the matrix of second derivatives evaluated at $\hat{\theta}_{i-1}$. The formulas for the first and second derivatives can be found in Seegrist and Arner's forthcoming publication.

There are two reasons for having different iterative procedures in the same program. One reason is that there are differences in the rates of convergence. Close to the solution, the Newton-Raphson procedure converges faster than the Fisher scoring procedure. If the estimates are away from the solution, the scoring procedure usually converges faster. The second reason is that the matrix of second derivatives, which is used in the Newton-Raphson procedure, may not be negative definite and its inverse cannot be determined. However, it may be that the estimate of V has an inverse, in which case, Q_{i-1} for the Fisher scoring procedure can be inverted and new estimates of θ can be calculated.

Following the suggestion of Jennrich and Sampson (1976), the Fisher scoring procedure is used for the first iteration and for any iteration for which the change in lnL at the preceding iteration is greater than a specified value. Also, the program switches to the Fisher scoring procedure if the Newton-Raphson procedure does not work.

At any iteration, new estimates for θ may result in a negative change in lnL. Let $\Delta_i = Q^{-1}{}_{i-1} S_{i-1}$ be the vector of increments to θ_{i-1} at iteration i. When Q_{i-1} is negative definite, an increase in lnL can be obtained with a sufficiently small portion of Δ_i, which we determine using a partial stepping procedure similar to that suggested by Jennrich and Sampson (1976). At each step, Δ_i is divided by 4 and new estimates of θ are calculated. The partial stepping is terminated when the value of $\ln L_i$ is more than the value of $\ln L_{i-1}$.

The iterations are terminated when $\ln L_i - \ln L_{i-1} < \epsilon$ where ϵ is a small predetermined value.

PROGRAM INFORMATION

The instructions for input cards are found in Appendix I. The Appendix explains the array sizes and instructions for changing the array sizes for problems that exceed the programed dimensions. The appendix also explains the error messages that are printed if a matrix cannot be inverted.

The program output includes iteration number, solution procedure used, and values of lnL, Q, β, and V. At the final iteration, the program prints R which is the coveriance matrix in correlation form, and prints $S(\hat{\beta})$ which is the sample covariance matrix of the estimated regression coefficients. The estimates of β and V are punched every 10 iterations and after the final iteration. The punched cards can be read as starting values for further iterations, if necessary.

Using the diagonal elements of $S(\hat{\beta})$ as the sample variance of the elements of $\hat{\beta}$, confidence intervals for the individual elements can be calculated.

Likelihood ratio tests can be used to test hypotheses about β for the composite vector of correlated variates. Tests of the type that a portion of the parameter vector is zero are obtained by running the program twice. The first run is with all of the independent variables. The second run is with the independent variables associated with the hypothesized zero portion of β dropped from the vector of independent variables.

If L_Ω is the value of the likelihood function under the full model with q_Ω independent variables and L_ω is the value of the likelihood function under the model restricted by the hypothesis which has q_ω independent variables, then $-2 \ln\lambda = -2 (\ln L_\omega - \ln L_\Omega)$ is distributed asymptotically as χ^2 with $q_\Omega - q_\omega$ degrees of freedom.

Other hypotheses can be tested with appropriate coding of the design matrix.

SAMPLE PROBLEMS

To demonstrate the program, we present two examples. The first example is based on Clutter's (1972) growth and yield model with incomplete repeated measurements on permanent plots. The second example has incomplete measurements because one of the characteristics can only be measured by destructive sampling.

We write Clutter's growth and yield model as

$$\mathcal{E}(\ln V_{ti} \mid \ln B_{ti}) = \beta_0 + \beta_1 S_i + \beta_2 A^{-1}{}_{ti} + \beta_3 \ln B_{ti}$$

$$\mathcal{E}(\ln B_{ti}) = A_{1i} A^{-1}{}_{ti} \ln B_{1i} + \alpha_1 (1 - A_{1i} A^{-1}{}_{ti}) + \alpha_2 (1 - A_{1i} A^{-1}{}_{ti}) S_i$$

where

V_{ti} = cubic foot volume per acre on the i^{th} plot at time t,

S_i = site index of i^{th} plot (in feet),

A_{ti} = stand age of i^{th} plot at time t,

and

B_{ti} = basal area per acre of the i^{th} plot at time t (in square feet).

We assume that $\ln V_{ti}$ and $\ln B_{ti}$ have a joint normal distribution. Therefore, the marginal distribution of $\ln B_{ti}$ and the conditional distribution of $\ln V_{ti}$ given $\ln B_{ti}$ are independent. The solution to the volume equation and basal area equation could be obtained separately, or they can be obtained in one pass by assigning the observations in the volume model and the observations in the basal area model to two different groups. Further grouping of the data occurs because measurements on some of the plots are incomplete. Of the 51 plots measured, 15 plots were measured 3 times; 16 plots were measured in the first and second period; the remaining 20 plots were measured in the first and third periods.

To obtain a solution to the volume and basal area equations in one pass, one instructs the program that there are six groups.

The data cards for the six groups have the following general form:

Variate Number	y	x_1	x_2	x_3	x_4	x_5	x_6

The data cards for plots in *GROUP 1* have the form

1	$\ln V_1$	1	S	A^{-1}_1	$\ln B_1$	0	0
2	$\ln V_2$	1	S	A^{-1}_2	$\ln B_2$	0	0
3	$\ln V_3$	1	S	A^{-1}_3	$\ln B_3$	0	0

The data cards for plots in *GROUP 2* have the form

4	$\ln B^*_2$	0	0	0	0	$1\text{-}A_1 A^{-1}_2$	$(1\text{-}A_1 A^{-1}_2)S$
5	$\ln B^*_3$	0	0	0	0	$1\text{-}A_1 A^{-1}_3$	$(1\text{-}A_1 A^{-1}_3)S$

The data cards for plots in *GROUP 3* have the form

1	$\ln V_1$	1	S	A^{-1}_1	$\ln B_1$	0	0
2	$\ln V_2$	1	S	A^{-1}_2	$\ln B_2$	0	0

The data cards for plots in *GROUP 4* have the form

4	$\ln B^*_2$	0	0	0	0	$1\text{-}A_1 A^{-1}_2$	$(1\text{-}A_1 A^{-1}_2)S$

The data cards for plots in *GROUP 5* have the form

1	$\ln V_1$	1	S	A^{-1}_1	$\ln B_1$	0	0
3	$\ln V_3$	1	S	A^{-1}_3	$\ln B_3$	0	0

The data cards for plots in *GROUP 6* have the form

5	$\ln B^*_3$	0	0		0	$1\text{-}A_1 A^{-1}_3$	$(1\text{-}A_1 A^{-1}_3)S$

where $\ln B^*_t$ is $\ln B_t - A_1 A^{-1}_t \ln B_1$.

The input cards for this example are as follows:

Control Card 1 — General Title Card

Field (1) — Title

Control Card 2 — Job Control Card

Field (1) — Blank DELT = 1.0×10^{-10}
Field (2) — Blank SNR = .001
Field (3) — 5 = number of variates
Field (4) — 6 = number of independent variables
Field (5) — 6 = number of groups
Field (6) — 40 = maximum number of iterations
Field (7) — 0 = results printed on last iteration only
Field (8) — 5 = data is located on file 5
Field (9) — 1 = number of variable format cards
Field (10) — 0 = no intercept in model
Field (11) — 0 = no unobserved pairs of variates
Field (12) — 0 = initial estimates to be calculated from data
Field (13) — 11 = number of comment cards to read and print

Control Card 3 — Group Variate Card

Field (1) — 3 = number of variates in group 1
Field (2) — 2 = number of variates in group 2
Field (3) — 2 = number of variates in group 3
Field (4) — 1 = number of variates in group 4
Field (5) — 2 = number of variates in group 5
Field (6) — 1 = number of variates in group 6

Control Card 4 — Group Sample Size Card

Field (1) — 16 = number of plots in group 1
Field (2) — 16 = number of plots in group 2
Field (3) — 15 = number of plots in group 3
Field (4) — 15 = number of plots in group 4
Field (5) — 20 = number of plots in group 5
Field (6) — 20 = number of plots in group 6

11 *Comment Cards*

Variable Format Card

185 *Data Cards*

The printed output consists of the information on the general title card, the job control values, the comment cards, and the variable format for the data.

This is followed by values of the initial estimates which are $\ln L_o$, Q_o, the vector $\hat{\beta}_o$, and the matrix $\hat{\Sigma}_o$.

The initial estimates are followed by the maximum likelihood estimates from the last iteration. The values of lnL and Q are given. This is followed by the maximum likelihood estimates $\hat{\beta}$, $\hat{\Sigma}$, and $\hat{R}$, and the sample covariance matrix of $\hat{\beta}$, $S(\hat{\beta})$.

For example, the maximum likelihood estimates are

Parameters	*Estimates* $(\hat{\beta})$	*Variance* $(s^2(\hat{\beta}))$
β_0	3.2924	0.0016
β_1	0.0117	0.1224×10^{-6}
β_2	-26.3890	0.4292
β_3	0.9665	0.3613×10^{-4}
α_1	4.0136	0.9978
α_2	0.0314	0.2225×10^{-3}

where $s^2(\hat{\beta})$ are the elements on the diagonal of the matrix $S(\hat{\beta})$ at iteration 13.

In the second example, we analyze an incomplete set of measurements which might occur in forestry. Suppose we have an experiment where two levels of fertilizer are applied to a sample of n tree seedlings. The first year height (H_1) is measured on all seedlings, and dry weight (W_1) is measured on m of the seedlings. The remaining seedlings are grown for one more year and height and dry weight are measured.

The experimental design is a one-way layout with incomplete multivariate measurements.

Altogether, four variates are measured; but because dry weight is a destructive measurement, it is not possible to observe the variate pairs (W_1, W_2) and (W_1, H_2). For convenience, we denote the v^{th} dependent variate measured on a seedling in the k^{th} group as Y_{kv}. The two variates in group 1 are $Y_{11} = H_1$ and $Y_{13} = W_1$. Similarly, the three variates measured on the seedlings in group 2 are $Y_{21} = H_1$, $Y_{22} = H_2$, and $Y_{24} = W_2$.

The model for the one-way layout can be written as

$$y_{kvij} = \mu_v + \tau_{vi} + \epsilon_{kvij};$$

where

y_{kvij} = the observed value of variate Y_v measured on the j^{th} seedling in the k^{th} group treated with the i^{th} fertilizer level,

and

ϵ_{kvij} = error of variate Y_v measured on j^{th} seedling in k^{th} group treated with the i^{th} fertilizer level.

Let μ_{vi} = the mean of the i^{th} treatment for the v^{th} variable;

then

$$\mu_v = (\mu_{v1} + \mu_{v2})/2$$

and

$$\tau_{vi} = \mu_{vi} - \mu_v.$$

Observational equations describing the two measurements on the j^{th} seedling in group 1 receiving fertilizer level i are

$$\begin{bmatrix} h_{1ij} \\ w_{1ij} \end{bmatrix} = \begin{bmatrix} 1 & \delta_{ij} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \delta_{ij} & 0 & 0 \end{bmatrix} \beta$$

where δ_{ij} is the dummy variable for the treatment effect and is

$$\delta_{ij} = 1 \text{ for } i = 1$$

$$= -1 \text{ for } i = 2$$

and the vector of regression coefficients β is

$$[\mu_1 \ \tau_1 \ \mu_2 \ \tau_2 \ \mu_3 \ \tau_3 \ \mu_4 \ \tau_4]'.$$

Similarly, the three observational equations for the j^{th} seedling in group 2 receiving treatment i would be

$$\begin{bmatrix} h_{1ij} \\ h_{2ij} \\ w_{2ij} \end{bmatrix} = \begin{bmatrix} 1 & \delta_{ij} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \delta_{ij} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \delta_{ij} \end{bmatrix} \beta.$$

Using the fictitious data given in Appendix II, we obtained the following estimates:

Variate	*Parameter*	*Estimate* $(\hat{\beta})$	*Variance* $(s^2(\hat{\beta}))$
H_1	μ_1	4.925	.0681
	τ_1	-0.475	.0681
H_2	μ_2	9.419	.1088
	τ_2	-0.922	.1088
W_1	μ_3	37.705	2.3103
	τ_3	-4.885	2.3103
W_2	μ_4	44.511	3.8375
	τ_4	-6.369	3.8375

To test the composite hypothesis of "no treatment effect," we need the value of the logarithm of the likelihood function which is $\ln L = -97.7$.

The hypothesis of "no treatment effect" for any of the responses is $\tau_i = 0$ for $i = 1, \ldots, 4$. The design matrix for the model under the hypothesis is obtained from the design matrix for the full model by dropping the four treatment dummy variables.

The value of lnL calculated when the restricted model was run was -104.2. The test statistic -2(-104.2 + 97.7) = 13.0 would be statistically significant ($\chi^2_{.05,4} = 9.47$).

The sample correlation matrix is printed out as

Variate	H_1	H_2	W_1	W_2
H_1	1.0			
H_2	0.79	1.0		
W_1	0.83	0.0	1.0	
W_2	0.68	.80	0.0	1.0

Notice that the correlations $r(W_1, W_2)$ and $r(W_1, H_2)$, are shown as having values 0.0. However, $r(W_1, W_2)$ and $r(W_1, H_2)$ are not estimable in this example.

This program can be used to obtain a solution when the measurements on all experimental units are complete. However, for the complete measurement situation, a computationally more efficient procedure is presented in Arner and Seegrist (1979). In fact, if only a portion of the data are complete, it may be better to use the first program on the complete subset to obtain estimates of β and Σ to use as initial estimates for the incomplete program.

The program was written in Fortran for an IBM 370/168. A deck and listing of the computer program described in this publication is available on request with the understanding that the U.S. Department of Agriculture cannot assure accuracy, completeness, reliability, or suitability for any other purpose than that reported. The recipient may not assert any proprietary rights thereto nor represent it to anyone as other than a Government-produced computer program. For cost information write U.S. Forest Service, 370 Reed Road, Broomall, Pa. 19008.

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APPENDIX I

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C **********************************************************************
C *** CONTROL CARDS
C   1. TITLE CARD   -    FORMAT = (10A8)
C   2. JOB CONTROL CARD
C      COLS.  1-10 DELT = END ITERATIONS VALUE. DEFAULT = 1.E-10, WHEN
C                         THE PROPORTIONATE CHANGE OF THE LOG LIKELIHOOD
C                         FUNCTION IS LESS THAN DELT, ITERATIONS STOP.
C            11-20 SNR  = NEWTON-RAPHSON, FISHER SCORING CUTOFF POINT.
C                         DEFAULT = .001, WHEN PROPORTIONATE CHANGE IN
C                         THE LOG LIKELIHOOD FUNCTION IS LESS THAN *SNR*
C                         THE NEWTON RAPHSON PROCEDURE IS USED;
C                         IF GREATER, THE FISHER PROCEDURE IS USED.
C                         NOTE: BOTH DELT AND SNR ARE READ WITH F10.3
C                         FORMAT.
C            21-22 P    = LENGTH OF Y. THE NO. OF DEPENDENT VARIATES,
C                         INCLUDING REPEATED MEASUREMENTS. NOT ALL P
C                         VARIATES NEED TO BE MEASURED ON EACH
C                         EXPERIMENTAL UNIT.
C            23-24 Q    = LENGTH OF X. THE NO. OF INDEPENDENT VARIABLES,
C                         THE INDEPENDENT VARIABLES CAN BE THE SAME OR
C                         DIFFERENT FOR EACH DEPENDENT VARIATE. WHERE
C                         AN INDEPENDENT VARIABLE IS NOT MEASURED FOR A
C                         PARTICULAR Y, A ZERO SHOULD BE CODED FOR THAT
C                         X.
C            25-26 NG   = NO. OF GROUPS. A GROUP IS DEFINED AS THE SET
C                         OF EXPERIMENTAL UNITS, EACH OF WHICH CONTAINS
C                         THE SAME SUBSET OF DEPENDENT VARIATES.
C            27-28 NITR = MAXIMUM NO. OF ITERATIONS (DEFAULT = 30).
C            29-30 IPR  = PRINT OPTION.    IF IPR = ....
C                       = 0 PRINT RESULTS AT LAST ITERATION ONLY.
C                       = 1 PRINT LOG LIKELIHOOD FUNCTION, DIFFERENCE,
C                           AND VALUE OF QUADRATIC FORM EACH ITERATION.
C                       = 2 PRINT ABOVE PLUS ESTIMATES OF BETA AND SIGMA
C                           EACH ITERATION.
C            31-32 DF   = FILE ON WHICH DATA IS LOCATED (DEFAULT = 5).
C                         IF THE DATA  ARE NOT ON  CARDS, THE FILE ON
C                         WHICH THE DATA RESIDES MUST BE DEFINED WITH
C                         APPROPRIATE JOB CONTROL LANGUAGE.
C            33-34 NF   = NO. OF FORMAT CARDS (MAX. = 5, DEFAULT = 1).
C            35-36 INT  = INTERCEPT INDICATOR; IF INT NOT EQUAL 0, Q IS
C                         INCREASED BY 1 AND A COLUMN OF 1'S IS ADDED
C                         TO THE X MATRIX.
C            37-38 MIS  = NO. OF PAIRS OF VARIATES NEVER OBSERVED
C                         TOGETHER ON THE SAME EXPERIMENTAL UNIT; IF Y1
C                         AND Y2 ARE NEVER OBSERVED TOGETHER, NO
C                         COVARIANCE BETWEEN THE TWO CAN BE CALCULATED.
C                         THE PROGRAM CAN DETERMINE THIS VALUE. HOWEVER
C                         IF IT IS KNOWN AND READ HERE, THE SIZE OF
C                         CERTAIN ARRAYS CAN BE REDUCED (SEE ARRAY SIZE
C                         DEFINITIONS BELOW).
C            39-40 INP  = INITIAL ESTIMATE INDICATOR;  IF INP = ...
C                       = 0 INITIAL ESTIMATES OF BETA AND SIGMA ARE
C                           CALCULATED FROM DATA.
C                       = 1 READ INITIAL VALUES FOR BETA AND SIGMA.
C                           (CARD TYPE 6 AND 7) MAY BE FROM PREVIOUS
C                           RUN.
C            41-42 NC   = NO. OF ADDITIONAL COMMENTS CARDS(CARD 5)
```

```
C    3. (NP(G),G=1,NG) , THE NO. OF DEPENDENT VARIATES FOR EACH GROUP
C                          FORMAT = (40I2).
C    4. (NPLT(G),G=1,NG),THE NO. OF EXPERIMENTAL UNITS (PLOTS) FOR EACH
C                          GROUP,    FORMAT = (16I5)
C    5. COMMENTS CARDS - OPTIONAL; READ ONLY IF *NC* ON CARD 2 IS > 0
C                          FORMAT = (10A8) -VARIABLE NAMES CAN BE READ HERE
C ***  READ CARD TYPES 6 AND 7 ONLY IF INP = 1,  FORMAT = (5D16.8).
C    6. INITIAL ESTIMATES OF BETA.
C    7. INITIAL ESTIMATES OF SIGMA.
C    8. FORMAT OF DATA  ( INCLUDING 1 INTEGER AS THE FIRST ITEM FOLLOWED
C          BY   Q+1  REAL FORMAT CODES)
C    9. DATA - THE DATA SHOULD BE CODED AS FOLLOWS:
C          A. THE FIRST POSITION OF EACH RECORD SHOULD HAVE AN INTEGER
C             VALUE SPECIFYING THE DEPENDENT VARIATE FOR THIS RECORD.
C.            THIS WILL BE A VALUE FROM  1 TO P.
C          B. THE DEPENDENT VARIATE SHOULD BE CODED NEXT.
C          C. THE Q INDEPENDENT VARIABLES FOLLOW THE DEPENDENT VARIATE.
C             THERE ARE Q POSITIONS FOR THE INDEPENDENT VARIABLES. NOT ALL
C             INDEPENDENT VARIABLES NEED TO BE OBSERVED WITH EACH
C             DEPENDENT VARIATE; A  0  SHOULD BE CODED IN THE APPROPRIATE
C             POSITION.
C          D. ALL CORRELATED OBSERVATIONS FOR AN EXPERIMENTAL UNIT (PLOT)
C             SHOULD BE CONTIGUUUS.
C          E. THE EXPERIMENTAL UNITS FOR EACH GROUP SHOULD BE CONTIGUOUS.
C ***********************************************************************
C *** EXAMPLE OF JCL FOR DEFINING DATA ON DISK WHERE DF=8 IN COLUMN 32
C     OF CONTROL CARD 2:
C     //GO.FT08F001 DD UNIT=3330,VOL=SER=UE1014,DSN=YIELD,DISP=SHR
C ***********************************************************************

C ***********************************************************************
C *** INSTRUCTIONS FOR CHANGING ARRAY DIMENSIONS. REQUIRED ONLY IN MAIN
C     ROUTINE.
C     1. DETERMINE NG,Q,P, AND NP(G),G=1,NG.
C     2. WITH THESE VALUES CALCULATE QQ,PP,QPPI,PPS,PSQS,PPI,PQP,PPP
C        USING ARRAY SIZE DEFINITIONS BELOW.
C     3. CHANGE DIMENSION SIZE OF ARRAYS TO APPROPRIATE VALUES,
C     4. CHANG MXC TO Q, MXXP TO P, MXQPPI TO QPPI, MXPPS TO PPS,
C        MXPSQS TO PSQS, MXPP TO PP, MXPPI TO PPI, MXNG TO NG,
C        MXNF TO NF.
C ***********************************************************************
C ***********************************************************************
C *** ARRAY SIZE DEFINITIONS
C     QQ  = C*(Q+1)/2                                        MAX = MXQQ
C     PP  = P*(P+1)/2                                        MAX = MXPP
C     PPI = PP-MIS   MIS IS DEFINED ON  JOB CONTROL CARD.    MAX = MXPPI
C     QPPI= C+PPI                                            MAX = MXQPPI
C     PPP = PPI*(PPI+1)/2
C     PQP = QPPI*(QPPI+1)/2                                  MAX = MXPQP
C     PPS = THE SUM OVER ALL GROUPS OF (NP(G)*(NP(G)+1)/2)   MAX = MXPPS
C     PSQS= THE SUM OVER ALL GROUPS OF (NP(G)*NP(G))
C *** ARRAY DIMENSIONS MUST BE AT LEAST THE SIZE BELOW.
C     REAL*8 SIGMA(PP),YPY(PPS),PLSS(PPP),Y(P),SIGINV(PPS),PLS(PPI),
C            DERIV1(QPPI),XPY(PSQS,Q),XPX(PPS,Q,Q),DELTA(QPPI),X(P,Q),
C            XPY2(Q),XPXB(Q),BETA(Q),PLBB(QQ),PLB(Q),
C            PLBS(PPI,Q),DERIV2(PQP)
C     INTEGER TP(NG,P),NP(NG),NPLT(NG),PTYPY(NG),PTXPY(NG),DROP(PP)
C ***********************************************************************
C ***********************************************************************
C     THERE ARE 5 PLACES WHERE FATAL MATRIX INVERSIONS MIGHT OCCUR. A
C     MESSAGE IS PRINTED WITH A NUMBER INDICATING WHERE THE INVERSION
C     FAILED.
C     1 = X'X
C     2 = INITIAL SIGMA OR PORTION OF SIGMA
C     3 = MATRIX OF EXPECTED VALUES OF SECOND DERIVATIVES WITH RESPECT
C        TO SIGMA.
C     4 = MATRIX OF SECOND DERIVATIVES WITH RESPECT TO BETA.
C     5 = GROUP PORTION OF NEW ESTIMATE OF SIGMA.
C ***********************************************************************
```

APPENDIX II

```
SAMPLE PROBLEM 1, GROWTH AND YIELD USING CLUTTER'S EQUATIONS, 6 GROUPS
                      5 6 640 0 5 1 0 0 011
 3 2 2 1 2 1
   16   16   15   15   20   20
*************************************************************************
VARIABLE NAMES
   Y1 = LN(VCL) AT TIME 1;  Y2 = LN(VOL) AT TIME 2;
   Y3 = LN(VOL) AT TIME 3;  Y4 = (LN(BA2) - LN(BA1)*AGE1/AGE2);
   Y5 = (LN(BA3) - LN(BA1)*AGE1/AGE3);  X1 = INTERCEPT; X2 = SITE INDEX;
   X3 = 1/(PLOT AGE); X4 = LN(BA(T)); X5 = (1-A(1)/A(T)); X6 = X5 * SI;
***   Y1,Y2,Y3 = F(X1,X2,X3,X4);   Y4,Y5 = F(X5,X6)   ***

  G  P  V     Y        X1    X2      X3         X4          X5         X6

*************************************************************************
(6X,I3,F10.5,2F5.1,4F10.5)
  1  1  1    7.92039  1.0 68.0   0.02041    4.50443    0.0        0.0
  1  1  2    8.02682  1.0 68.0   0.01818    4.56226    0.0        0.0
  1  1  3    8.14159  1.0 68.0   0.01695    4.62473    0.0        0.0
  1  2  1    7.69006  1.0 72.0   0.02273    4.29007    0.0        0.0
  1  2  2    7.88382  1.0 72.0   0.02000    4.42700    0.0        0.0
  1  2  3    8.04967  1.0 72.0   0.01852    4.53828    0.0        0.0
                                   .
                                   .
                                   .
  2  1  4    0.54923  0.0  0.0   0.0        0.0        0.10909    7.41818
  2  1  5    0.88376  0.0  0.0   0.0        0.0        0.16949   11.52542
  2  2  4    0.65174  0.0  0.0   0.0        0.0        0.12000    8.64000
  2  2  5    1.04267  0.0  0.0   0.0        0.0        0.18519   13.33333
                                   .
                                   .
                                   .
  3  1  1    6.69724  1.0 61.0   0.02500    3.45526    0.0        0.0
  3  1  2    7.03527  1.0 61.0   0.02174    3.72569    0.0        0.0
  3  2  1    7.65128  1.0 75.0   0.02000    4.19750    0.0        0.0
  3  2  2    7.79770  1.0 75.0   0.01887    4.28110    0.0        0.0
                                   .
                                   .
                                   .
  4  1  4    0.72112  0.0  0.0   0.0        0.0        0.13043    7.95652
  4  2  4    0.32119  0.0  0.0   0.0        0.0        0.05660    4.24528
                                   .
                                   .
                                   .
  5  1  1    7.54987  1.0 74.0   0.03125    4.26637    0.0        0.0
  5  1  3    7.78030  1.0 74.0   0.02564    4.41502    0.0        0.0
  5  2  1    7.10497  1.0 77.0   0.03030    3.81777    0.0        0.0
  5  2  3    7.69454  1.0 77.0   0.02500    4.28827    0.0        0.0
                                   .
                                   .
                                   .
  6  1  5    0.91441  0.0  0.0   0.0        0.0        0.17949   13.28205
  6  2  5    1.13861  0.0  0.0   0.0        0.0        0.17500   13.47500
```

```
SAMPLE PROBLEM 1, GROWTH AND YIELD USING CLUTTER'S EQUATIONS, 6 GROUPS

PROGRAM CONTROL INFORMATION
******************************
P  (NO.  OF VARIATES, INCLUDING DIFFERENT VARIABLES AND REPEATED MEASUREMENTS) =      5
Q  (NO. OF INDEPENDENT VARIABLES) =                                                   6
NG  (NO. OF GROUPS) =                                                                 6
NITR  (MAXIMUM NO. OF ITERATIONS) =                                                  40
IPR  (PRINT OPTION) =                                                                 0
DF  (FILE ON WHICH DATA IS LOCATED) =                                                 5
NF  (NO.OF FORMAT CARDS) =                                                            1
INP  (INITIAL VALUES INDICATOR) =                                                     0
MIS  (NO. OF MISSING VARIATE COMBINATICNS) =                                          0
INT  (INTERCEPT INDICATOR) =                                                          0
DELT  (END ITERATIONS VALUE) =                                                0.100E-09
SNR  (NEWTON RAPHSON - FISHER SCORING   CUTOFF POINT) =                       0.100E-02

NO. OF VARIATES (P) AND EXPERIMENTAL UNITS OR PLOTS (N) MEASURED IN EACH GROUP
*****************************************************************************
GROUP    1    2    3    4    5    6
P(G)     3    2    2    1    2    1
N(G)    16   16   15   15   20   20

ADDITIONAL PROBLEM OR VARIABLE IDENTIFICATION
************************************************************************
************************************************************************
VARIABLE NAMES
   Y1 = LN(VOL) AT TIME 1;  Y2 = LN(VOL) AT TIME 2;
   Y3 = LN(VOL) AT TIME 3;  Y4 = (LN(BA2) - LN(BA1)*AGE1/AGE2);
   Y5 = (LN(BA3) - LN(BA1)*AGE1/AGE3);  X1 = INTERCEPT; X2 = SITE INDEX;
   X3 = 1/(PLOT AGE); X4 = LN(BA(T)); X5 = (1-A(1)/A(T)); X6 = X5 * SI;
***   Y1,Y2,Y3 = F(X1,X2,X3,X4);   Y4,Y5 = F(X5,X6)   ***

  G  P  V    Y       X1   X2     X3        X4        X5        X6

************************************************************************

FORMAT OF DATA
***************
(6X,I3,F10.5,2F5.1,4F10.5)

INITIAL VALUES
***************

LOG OF THE LIKLIHOOD FUNCTION =    220.78277
************************************************

THE VALUE OF THE QUADRATIC FORM =     492.19537
***************************************************

INITIAL ESTIMATE OF REGRESSION COEFFICIENTS  (BETA)
*****************************************************
  0.33769660D+01  0.10295165D-01 -0.26387855D+02  0.97374591D+00  0.46237813D+01

  0.17473433D-01
INITIAL ESTIMATE OF SIGMA
*************************

  1   0.89295320-03
  2   0.12004840-03   0.14363440-03
  3   0.21001150-03   0.44591300-04   0.73366720-03
  4   0.0             0.0             0.0             0.27375800-02
  5   0.0             0.0             0.0             0.74245960-03   0.48651280-02
```

```
ITERATION  =    13      CRITERION USED AT THIS STEP = NEWTON RAPHSON
*****************************************************************

LOG OF THE LIKLIHOOD FUNCTION =    318.09751        DIFFERENCE =    .66631856D-09
*****************************************************************************************
PROPORTIONATE DIFFERENCE =    .20946985L-11

VALUE OF THE QUADRATIC FORM =     185.00000
*********************************************

NO. OF QUARTERING STEPS USED =    0
**********************************

REGRESSION COEFFICIENTS  (BETA)
*******************************
  0.32924017D+01  0.11682717D-01 -0.26388781D+02  0.96652491D+00  0.40136489D+01  0.31409962D-01

ESTIMATE OF COVARIANCE MATRIX  (SIGMA)
**************************************
  1   0.19484190-02
  2   0.46803600-03   0.22831480-03
  3   0.15533210-02   0.74079890-03   0.35864010-02
  4   0.0             0.0             0.0             0.12541080-01
  5   0.0             0.0             0.0             0.12929080-01   0.14226530-01

CORRELATIONS
*************
  1   0.10000000+01
  2   0.70173170+00   0.10000000+01
  3   0.58761180+00   0.81866060+00   0.10000000+01
  4   0.0             0.0             0.0             0.10000000+01
  5   0.0             0.0             0.0             0.96792190+00   0.10000000+01

SAMPLE COVARIANCE MATRIX OF ESTIMATED REGRESSION COEFFICIENTS
**************************************************************
  1   0.15790950-02
  2  -0.83163860-05   0.12240100-06
  3  -0.14642300-01  -0.45973600-06   0.42919240+00
  4  -0.17541830-03   0.18563740-07   0.13821620-02   0.36135970-04
  5   0.0             0.0             0.0             0.0             0.99776310+00
  6   0.0             0.0             0.0             0.0            -0.14836520-01   0.22249410-03
```

```
SAMPLE PROBLEM 2, FULL MODEL MANOVA,  2 GROUPS, 2 TREATMENTS
                      4 8 220 0 5 1 0 0 010
 2 3
   10    10
*****************************************************************************
VARIABLE NAMES
   V  = VARIATE NUMBER
   Y1 = HEIGHT INCREASE FOR FIRST YEAR
   Y2 = HEIGHT INCREASE FOR SECOND YEAR
   Y3 = DRY WEIGHT FOR FIRST YEAR
   Y4 = DRY WEIGHT FOR SECOND YEAR
   X = X1=X2=X3=X4 = DUMMY VARIABLES FOR INTERCEPT AND TREATMENTS
 V      Y       X1     X2     X3     X4
*****************************************************************************
(I2,F7.1,3X,4(2F2.0,2X))
 1     4.8     1 1    0 0    0 0    0 0      GROUP 1, TREATMENT 1
 3    31.0     0 0    0 0    1 1    0 0
 1     4.6     1 1    0 0    0 0    0 0
 3    41.0     0 0    0 0    1 1    0 0
 1     5.8     1 1    0 0    0 0    0 0
 3    35.0     0 0    0 0    1 1    0 0
 1     4.3     1 1    0 0    0 0    0 0
 3    33.0     0 0    0 0    1 1    0 0
 1     3.9     1 1    0 0    0 0    0 0
 3    29.0     0 0    0 0    1 1    0 0
 1     4.9     1-1    0 0    0 0    0 0      GROUP 1, TREATMENT 2
 3    37.0     0 0    0 0    1-1    0 0
 1     5.3     1-1    0 0    0 0    0 0
 3    42.0     0 0    0 0    1-1    0 0
 1     4.7     1-1    0 0    0 0    0 0
 3    40.0     0 0    0 0    1-1    0 0
 1     3.9     1-1    0 0    0 0    0 0
 3    36.0     0 0    0 0    1-1    0 0
 1     6.1     1-1    0 0    0 0    0 0
 3    49.0     0 0    0 0    1-1    0 0
 1     4.5     1 1    0 0    0 0    0 0      GROUP 2, TREATMENT 1
 2     8.9     0 0    1 1    0 0    0 0
 4    33.0     0 0    0 0    0 0    1 1
 1     3.2     1 1    0 0    0 0    0 0
 2     6.7     0 0    1 1    0 0    0 0
 4    41.0     0 0    0 0    0 0    1 1
 1     4.3     1 1    0 0    0 0    0 0
 2     9.0     0 0    1 1    0 0    0 0
 4    37.0     0 0    0 0    0 0    1 1
 1     2.4     1 1    0 0    0 0    0 0
 2     5.1     0 0    1 1    0 0    0 0
 4    22.0     0 0    0 0    0 0    1 1
 1     6.7     1 1    0 0    0 0    0 0
 2     9.8     0 0    1 1    0 0    0 0
 4    53.0     0 0    0 0    0 0    1 1
 1     8.1     1-1    0 0    0 0    0 0      GROUP 2, TREATMENT 2
 2    12.2     0 0    1-1    0 0    0 0
 4    55.0     0 0    0 0    0 0    1-1
 1     4.8     1-1    0 0    0 0    0 0
 2     9.6     0 0    1-1    0 0    0 0
 4    48.0     0 0    0 0    0 0    1-1
 1     5.2     1-1    0 0    0 0    0 0
 2    10.1     0 0    1-1    0 0    0 0
 4    49.0     0 0    0 0    0 0    1-1
 1     4.3     1-1    0 0    0 0    0 0
 2     9.5     0 0    1-1    0 0    0 0
 4    51.0     0 0    0 0    0 0    1-1
 1     6.7     1-1    0 0    0 0    0 0
 2    12.1     0 0    1-1    0 0    0 0
 4    60.0     0 0    0 0    0 0    1-1
SAMPLE PROBLEM 2, REDUCED MODEL,  2 GROUPS, 2 TREATMENTS
                      4 4 220 0 8 1 0 0 0 0
 2 3
   10    10
(I2,F7.1,3X,4(F2.0,4X))
```

```
SAMPLE PROBLEM 2, REDUCED MODEL,  2 GROUPS, 2 TREATMENTS

PROGRAM CONTROL INFORMATION
******************************
P  (NO.  OF VARIATES, INCLUDING DIFFERENT VARIABLES AND REPEATED MEASUREMENTS) =      4
Q  (NO. OF INDEPENDENT VARIABLES) =                                                   4
NG  (NO. OF GROUPS) =                                                                 2
NITR  (MAXIMUM NO. OF ITERATIONS) =                                                  20
IPR  (PRINT OPTION) =                                                                 0
DF  (FILE ON WHICH DATA IS LOCATED) =                                                 8
NF  (NO.OF FORMAT CARDS) =                                                            1
INP  (INITIAL VALUES INDICATOR) =                                                     0
MIS  (NO. OF MISSING VARIATE COMBINATIONS) =                                          0
INT  (INTERCEPT INDICATOR) =                                                          0
DELT  (END ITERATIONS VALUE) =                                                0.100E-09
SNR  (NEWTON RAPHSON - FISHER SCORING   CUTOFF POINT) =                       0.100E-02

NO. OF VARIATES (P) AND EXPERIMENTAL UNITS OR PLOTS (N) MEASURED IN EACH GROUP
*************************************************************************************
GROUP    1    2
P(G)     2    3
N(G)    10   10

FORMAT OF DATA
**************
(I2,F7.1,3X,4(F2.0,4X))

INITIAL VALUES
**************

LOG OF THE LIKLIHOOD FUNCTION =   -109.85307
*************************************************

THE VALUE OF THE QUADRATIC FORM =      73.049606
****************************************************

INITIAL ESTIMATE OF REGRESSION COEFFICIENTS  (BETA)
******************************************************
  0.49250000D+01  0.55000000D+01  0.37300000D+02  0.44900000D+02


INITIAL ESTIMATE OF SIGMA
************************

  1   0.15868750D+01
  2   0.13465000D+01   0.17560000D+01
  3   0.12505000D+01   0.0              0.15705000D+02
  4   0.72360000D+01   0.93350000D+01   0.0              0.60145000D+02
```

```
ITERATION  =    10      CRITERION USED AT THIS STEP = NEWTON RAPHSON
**********************************************************************

LOG OF THE LIKLIHOOD FUNCTION =   -104.24262          DIFFERENCE =    .11723955D-12
*****************************************************************************************
PROPORTIONATE DIFFERENCE = .11246795E-14
VALUE OF THE QUADRATIC FORM =     50.000000
************************************************

NO. OF QUARTERING STEPS USED =    0
*********************************

REGRESSION COEFFICIENTS  (BETA)
*******************************
  0.4925000D+01  0.9404167D+01  0.37738778D+02  0.4385001D+02

ESTIMATE OF COVARIANCE MATRIX  (SIGMA)
**************************************
  1   0.1586875D+01
  2   0.1600785D+01   0.2410211D+01
  3   0.8164522D+01   0.0             0.6054900D+02
  4   0.8602508D+01   0.1274906D+02   0.0             0.8847127D+02

CORRELATIONS
*************
  1   0.1000000D+01
  2   0.8185289D+00   0.1000000D+01
  3   0.8329251D+00   0.0             0.1000000D+01
  4   0.7260267D+00   0.8730709D+00   0.0             0.1000000D+01

SAMPLE COVARIANCE MATRIX OF ESTIMATED REGRESSION COEFFICIENTS
*************************************************************
  1   0.7934375D-01
  2   0.8003923D-01   0.1602803D+00
  3   0.4082261D+00   0.4118044D+00   0.3954564D+01
  4   0.4301254D+00   0.8410104D+00   0.2213009D+01   0.6515401D+01
```

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- Berea, Kentucky, in cooperation with Berea College.
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