A COMPUTER PROGRAM

for the Maximum Likelihood Estimator of the General Multivariate Linear Model with Correlated Errors

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Abstract

This report describes a computer program that obtains maximum likelihood estimates of the parameters for a general linear model with correlated observations. The variates can be a set of different types of variates, repeated measurement on the same variate, or combinations of repeated measurements and different variates. A forest growth and yield model describing repeated measurement of plot volume and basal area is presented as an example.

IN THIS REPORT we describe a computer program that obtains maximum likelihood estimates of the parameters for a general multivariate linear model with two or more correlated observations. The variates can be a set of different types of variates, repeated measurements on the same variate, or combinations of repeated measurements and different variates.

We present, as an example, the growth and yield model described by Clutter (1963). This model uses recursive simultaneous equations to describe plot volume and basal area where the values are measured repeatedly on the same plot.

The general multivariate linear model for p correlated measurements on each of n plots can be written

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \qquad (1)$$

where $\mathbf{y}' = (\mathbf{y}'_1, \mathbf{y}'_2, \ldots, \mathbf{y}'_p)$

$\mathbf{X}' = (\mathbf{X}'_1, \mathbf{X}'_2, \ldots, \mathbf{X}'_p)$

and $\mathbf{y}'_i = (y_{i1}, \ldots, y_{in})$

$\mathbf{X}'_i = (\mathbf{x}'_{i1}, \ldots, \mathbf{x}'_{in})$

$\mathbf{x}_{ij} = (x_{ij1}, \ldots, x_{ijq})$.

The element y_{ij} is the observed value of variate Y_i on plot j and x_{ijk} is the observed value of the k^{th} independent variable associated with Y_i on plot j.

We assume that the vectors of observations for the n plots are mutually independent and that the p observations for each plot have the covariance matrix $\boldsymbol{\Sigma}$. Therefore, the covariance matrix for y can be written as $\boldsymbol{\Psi} = \boldsymbol{\Sigma} \text{ X } \mathbf{I}$, where X represents the Kronecker product. The order of $\boldsymbol{\Sigma}$ is p, and I is the identity matrix of order n.

Due to correlation among the observations, the ordinary least squares procedure cannot be used to solve the system (1) for estimates of $\boldsymbol{\beta}$. Therefore, we developed a computer program to obtain maximum likelihood estimates for $\boldsymbol{\beta}$ and $\boldsymbol{\Sigma}$.

The program maximizes the logarithm of the likelihood function of y;

$$L = -np/2 \ln(2\Pi) + n/2|\boldsymbol{\Sigma}^{-1}| - 1/2\, Q \qquad (2)$$

where $Q = (\mathbf{y}-\mathbf{X}\boldsymbol{\beta})' \boldsymbol{\Psi}^{-1} (\mathbf{y}-\mathbf{X}\boldsymbol{\beta})$

and $|\boldsymbol{\Sigma}^{-1}|$ denotes the determinant of matrix $\boldsymbol{\Sigma}^{-1}$.

For the maximization of L, we require the first and second derivatives of L with respect to $\boldsymbol{\beta}$ and $\boldsymbol{\Sigma}^{-1}$. Denote the tu element of $\boldsymbol{\Sigma}^{-1}$ as σ^{tu}. The required derivatives are presented in a form convenient for computation. Their derivation can be found in Seegrist and Arner (1978).

The first derivatives can be written

$$\frac{\partial L}{\partial \boldsymbol{\beta}} = \sum_t \sigma^{tt}(\mathbf{X}'_t\mathbf{y}_t - \mathbf{X}'_t\mathbf{X}_t\boldsymbol{\beta}) + \sum_t \sum_{u>t} \sigma^{tu}[\mathbf{X}'_t\mathbf{y}_u + \mathbf{X}'_u\mathbf{y}_t - (\mathbf{X}'_t\mathbf{X}_u + \mathbf{X}'_u\mathbf{X}_t)\boldsymbol{\beta}] \tag{3}$$

$$\frac{\partial L}{\partial \sigma^{tt}} = \frac{n}{2}\sigma_{tt} - \frac{1}{2}(\mathbf{y}'_t\mathbf{y}_t - 2\boldsymbol{\beta}'\mathbf{X}'_t\mathbf{y}_t + \boldsymbol{\beta}'\mathbf{X}'_t\mathbf{X}_t\boldsymbol{\beta})$$

$$\frac{\partial L}{\partial \sigma^{tu}} = n\sigma_{tu} - (\mathbf{y}'_t\mathbf{y}_u - \boldsymbol{\beta}'\mathbf{X}'_t\mathbf{y}_u - \boldsymbol{\beta}'\mathbf{X}'_u\mathbf{y}_t + \boldsymbol{\beta}'\mathbf{X}'_t\mathbf{X}_u\boldsymbol{\beta}).$$

The second order derivatives are

$$\frac{\partial^2 L}{\partial \boldsymbol{\beta}\partial \boldsymbol{\beta}} = -\sum_t \sigma^{tt}\mathbf{X}'_t\mathbf{X}_t - \sum_t \sum_{t<u} \sigma^{tu}(\mathbf{X}'_t\mathbf{X}_u + \mathbf{X}'_u\mathbf{X}_t) \tag{4}$$

$$\frac{\partial^2 L}{\partial \sigma^{rs}\sigma^{tu}} = -n(\sigma_{tr}\sigma_{su} + \sigma_{ts}\sigma_{ru})$$

$$\frac{\partial^2 L}{\partial \sigma^{rs}\sigma^{tt}} = -n\,\sigma_{tr}\sigma_{st}$$

$$\frac{\partial^2 L}{\partial \sigma^{rr}\sigma^{tu}} = -n\,\sigma_{tr}\sigma_{ru}$$

$$\frac{\partial^2 L}{\partial \sigma^{rr}\sigma^{tt}} = -\frac{n}{2}\sigma_{tr}\sigma_{tr}$$

$$\frac{\partial^2 L}{\partial \boldsymbol{\beta}\partial \sigma^{tu}} = \mathbf{X}'_t\mathbf{y}_u + \mathbf{X}'_u\mathbf{y}_t - (\mathbf{X}'_t\mathbf{X}_u + \mathbf{X}'_u\mathbf{X}_t)\boldsymbol{\beta} \qquad \text{if } t \neq u$$

$$= \mathbf{X}'_t\mathbf{y}_t - \mathbf{X}'_t\mathbf{X}_t\boldsymbol{\beta} \qquad \text{if } t = u.$$

CALCULATING THE MAXIMUM LIKELIHOOD ESTIMATES

The likelihood equations are nonlinear and have to be solved by iterative procedures. The initial estimates of the model parameters that we use are the ordinary least squares estimates $\hat{\boldsymbol{\beta}}_0 = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$, and the estimated variances and covariances based on $\hat{\boldsymbol{\beta}}_0$

$$\hat{\sigma}_{tu0} = (\mathbf{y}'_t\mathbf{y}_u - \hat{\boldsymbol{\beta}}'_0\mathbf{X}'_t\mathbf{y}_u - \hat{\boldsymbol{\beta}}'_0\mathbf{X}'_u\mathbf{y}_t + \hat{\boldsymbol{\beta}}'_0\mathbf{X}'_t\mathbf{X}_u\hat{\boldsymbol{\beta}}_0)/N.$$

The program uses three procedures for solving the likelihood equations; (1) Newton-Raphson; (2) Fisher scoring; and (3) the iterated Aitken procedure. Different iterative procedures are needed because one of the procedures may not converge to a solution; and the different procedures converge at different rates. The method of determining which procedure to use at each iteration is: if the change in L is less than a specified value at the preceeding iteration, the Newton-Raphson procedure is used; if the change in L is greater than the specified value, or if the Newton-Raphson procedure does not work, the Fisher scoring method is used. Finally, the iterated Aitken method is used if both Newton-Raphson and Fisher scoring procedures fail to work. These two procedures may not work at a particular iteration because of an ill-conditioned matrix of second derivatives, an ill-conditioned estimate of Σ, or a decrease in the value of the likelihood function when evaluated at the new estimates.

The three methods of solving the likelihood equations are based on calculating a vector of increments $(\boldsymbol{\Delta})$ to the vector of parameter estimates $(\hat{\boldsymbol{\theta}})$. The iterated parameter estimates are $\hat{\boldsymbol{\theta}}_i = \hat{\boldsymbol{\theta}}_{i-1} + \boldsymbol{\Delta}_i$.

The vector of increments $\boldsymbol{\Delta}$ for the Newton-Raphson procedure is

$$\boldsymbol{\Delta}_i = -\begin{bmatrix} \dfrac{\partial^2 L}{\partial \boldsymbol{\beta}\partial \boldsymbol{\beta}} & \dfrac{\partial^2 L}{\partial \boldsymbol{\beta}\partial \boldsymbol{\Sigma}^{-1}} \\[2ex] \dfrac{\partial^2 L}{\partial \boldsymbol{\Sigma}^{-1}\partial \boldsymbol{\beta}} & \dfrac{\partial^2 L}{\partial \boldsymbol{\Sigma}^{-1}\partial \boldsymbol{\Sigma}^{-1}} \end{bmatrix}^{-1}_{(i-1)} \begin{bmatrix} \dfrac{\partial L}{\partial \boldsymbol{\beta}} \\[2ex] \dfrac{\partial L}{\partial \boldsymbol{\Sigma}^{-1}} \end{bmatrix}_{(i-1)} \tag{5}$$

The first matrix on the right side of (5) is the inverse of the matrix of second derivatives of the likelihood function and the second matrix is the vector of first derivatives. The notation $[\;]_{i-1}$ indicates that the elements of the vector of first derivatives (3) and the elements of the matrix of second derivatives (4) are evaluated at $\hat{\boldsymbol{\theta}}_{i-1}$. $\partial L/\partial \boldsymbol{\Sigma}^{-1}$ denotes a vector whose tuth element is $\partial L/\partial \sigma^{tu}$, $\partial^2 L/\partial \boldsymbol{\beta} \partial \boldsymbol{\Sigma}^{-1}$ is a matrix whose tuth column is the vector $\partial^2 L/\partial \boldsymbol{\beta} \partial \sigma^{tu}$, and $\partial^2 L/\partial \boldsymbol{\Sigma}^{-1} \partial \boldsymbol{\Sigma}^{-1}$ is a matrix whose element in row (r,s) and column (t,u) is $\partial^2 L/\partial \sigma^{rs} \partial \sigma^{tu}$, for $1 \leq r \leq s \leq 1, \ldots, p$, $1 \leq t \leq u = 1, \ldots, p$.

The vector of increments for the Fisher scoring procedure is

$$\boldsymbol{\Delta}_i = - \begin{bmatrix} \dfrac{\partial^2 L}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}} & \mathbf{0} \\ \mathbf{0} & \dfrac{\partial^2 L}{\partial \boldsymbol{\Sigma}^{-1} \partial \boldsymbol{\Sigma}^{-1}} \end{bmatrix}^{-1}_{(i-1)} \begin{bmatrix} \dfrac{\partial L}{\partial \boldsymbol{\beta}} \\ \dfrac{\partial L}{\partial \boldsymbol{\Sigma}^{-1}} \end{bmatrix}_{(i-1)} \tag{6}$$

The first step of the iterated Aitken procedure is to calculate the increment to the vector of regression coefficients ($\hat{\boldsymbol{\beta}}$) which is

$$\boldsymbol{\Delta}(\hat{\boldsymbol{\beta}}_i) = - \left[\frac{\partial^2 L}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}}\right]^{-1}_{(i-1)} \left[\frac{\partial L}{\partial \boldsymbol{\beta}}\right]_{(i-1)}$$

The iterated estimate of the vector of regression coefficients is $\hat{\boldsymbol{\beta}}_i = \hat{\boldsymbol{\beta}}_{i-1} + \boldsymbol{\Delta}(\hat{\boldsymbol{\beta}}_i)$. It can be shown that $\hat{\boldsymbol{\beta}}_i$ is the generalized least squares estimator of $\boldsymbol{\beta}$ evaluated at $\hat{\boldsymbol{\Sigma}}_{i-1}$. The iterated variance and covariance estimates are

$$\hat{\sigma}_{tui} = (\mathbf{y}'_t \mathbf{y}_u - \hat{\boldsymbol{\beta}}'_i (\mathbf{X}'_t \mathbf{y}_u + \mathbf{X}'_u \mathbf{y}_t) + \hat{\boldsymbol{\beta}}'_i \mathbf{X}'_t \mathbf{X}_u \hat{\boldsymbol{\beta}}_i)/N.$$

The iterations are terminated when $L_i - L_{i-1} < \epsilon$ where ϵ is a small predetermined value, and L_i is the logarithm of the likelihood function evaluated at $\hat{\boldsymbol{\theta}}_i$.

The major computational problem is the matrix inversion required for the calculation of $\boldsymbol{\Delta}_i$. For the Newton-Raphson procedure, the matrix of second derivatives evaluated at $\hat{\boldsymbol{\theta}}_{i-1}$ may be singular. We also found that, when the residuals are highly correlated, the matrix of second derivatives with respect to the elements of $\boldsymbol{\Sigma}^{-1}$ is ill-conditioned. This second problem was alleviated when we noticed that the inverse of $\partial^2 L/\partial \boldsymbol{\Sigma}^{-1} \partial \boldsymbol{\Sigma}^{-1}$ could be calculated directly from the estimates of σ^{ij} calculated at the previous iteration.

Denote $\partial^2 L/\partial \boldsymbol{\Sigma}^{-1} \partial \boldsymbol{\Sigma}^{-1}$ as $\boldsymbol{\Lambda}$. We found that the element in row (r,s) and column (t,u) of the inverse of $\boldsymbol{\Lambda}$ is

$$\Lambda^{(rs),(tu)} = -\frac{1}{n} (\sigma^{tr} \sigma^{su} + \sigma^{ts} \sigma^{ru}).$$

Thus, the Fisher scoring procedure requires only one matrix inversion, the qxq matrix of second derivatives with respect to $\boldsymbol{\beta}$. If the matrix of second derivatives needed for the Newton-Raphson procedure is negative definite, it's inverse can be calculated with one more inversion of a matrix of order q by using definitions of the inverse of a partitioned matrix.

The point at which an iteration should be started with the Newton-Raphson procedure has not been determined. In their procedure for variance component estimation, Jennrich and Sampson (1976) use the scoring algorithm whenever the change in L is greater than 1.0. On the data sets we have worked with, we have found that the Newton-Raphson procedure converges more rapidly than the other two procedures when the full matrix of second derivatives can be inverted. The point at which this occurs seems a characteristic of each data set.

When a solution is obtained by a particular algorithm, but the change in L is negative, the program switches to a partial stepping procedure similar to that suggested by Jennrich and Sampson (1976). At each step, $\boldsymbol{\Delta}$ is divided by 4 and new estimates are obtained. The partial stepping is terminated at the first step which increases L.

PROGRAM INFORMATION

The program arrays are dimensioned for a maximum p of 10 and a maximum q of 12. If larger problems are encountered, the size of the appropriate arrays should be changed. Instructions for doing this are found in the comments section at the beginning of the program. The control card instructions are also located here, as well as in Appendix I.

When a matrix cannot be inverted, the program terminates with an error message. Appendix I contains an explanation of these messages.

The program output includes the estimates of $\boldsymbol{\beta}$, $\boldsymbol{\Sigma}$, $|\hat{\boldsymbol{\Sigma}}|$, the value of L evaluated at these estimates, and the inverse of the negative of the matrix of second derivatives with respect to $\boldsymbol{\beta}$ and $\boldsymbol{\Sigma}$. The elements of these matrices can be used as estimates of the variances and covariances of $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\Sigma}}$.

The value of L can be used to set up likelihood ratio tests to determine whether some of the independent variables can be dropped from the model. If L_Ω is the value of the log likelihood function under the full model, and L_ω is the value obtained with the restricted model (deletion of a portion of $\boldsymbol{\beta}$ of length q_1), then $-2\ (L_\omega - L_\Omega)$ is asymptotically distributed as χ^2 with q_1 degrees of freedom.

The estimates of $\boldsymbol{\beta}$ and $\boldsymbol{\Sigma}$ are both printed and punched. The punched cards can be read as starting values for further iterations (see program control information).

PROGRAM APPLICATION

The program was initially developed to obtain solutions to the forest growth and yield functions as described by Sullivan and Clutter (1972) when there are more than two repeated measurements per plot. The model and procedure presented here are more general, however. With proper construction of the data, the program can obtain solutions to the usual multivariate linear regression model as well as solutions to some systems of equations, where the dependent variables are combinations of repeated measurements and measurements on several correlated variates. Note that the X_i can be combinations of experimental design variables and covariates and need not be the same for the different y_i.

A sample problem is presented in Appendix II for 3 repeated measurements on each of 22 fixed plots. We use Clutter's (1963) growth and yield model to describe plot volume and basal area. The equations can be written as

$$E(\ln V_{ti}) = \beta_0 + \beta_1 S_i + \beta_2 A_{ti}^{-1} + \beta_3 \ln B_{ti}$$

$$E(\ln B_{ti}) = A_{1i}A_{ti}^{-1} \ln B_{1i} + \alpha_1(1 - A_{1i}A_{ti}^{-1}) + \alpha_2(1 - A_{1i}A_{ti}^{-1})S_i$$

where V_{ti} = cubic foot volume per acre on the i^{th} plot at time t,
S_i = site index of i^{th} plot (in feet),
A_{ti} = stand age of i^{th} plot at time t,
and B_{ti} = basal area per acre of the i^{th} plot at time t (in square feet).

We consider stand ages (A_t) and initial basal area (B_1) as design variables. Note that the basal area equation for time 1 is an identity, while the coefficient for $A_{1i}/A_{ti}\ \ln B_{1i}$ in the basal area equation for the second and third remeasurement periods is always 1. To obtain the computing form for the data, we delete the first basal area equation and subtract $A_1A_t^{-1}\ \ln B_1$ from both sides of the t^{th} basal area equation for t = 2, 3.

Appendix II contains a partial listing of the sample data and the control cards required to run the program with the data. The first four fields of the sample data cards contain identification information only, and are not used by the program. The values are plot number, measurement period, equation type, and variate number.

The data for the five equations for each plot are together. The first three cards have the data for the three volume equations, the last two cards have the data for the basal area equations for time 2 and 3. The data on the data cards are the values of the variables defined as shown in Table 1.

Table 1.—Data layout

Y	X_1	X_2	X_3	X_4	X_5	X_6
$\ln V_1$	1.0	S	A_1^{-1}	$\ln B_1$	0.0	0.0
$\ln V_2$	1.0	S	A_2^{-1}	$\ln B_2$	0.0	0.0
$\ln V_3$	1.0	S	A_3^{-1}	$\ln B_3$	0.0	0.0
$\ln B_2^*$	0.0	0.0	0.0	0.0	$1 - A_1A_2^{-1}$	$(1 - A_1A_2^{-1})S$
$\ln B_3^*$	0.0	0.0	0.0	0.0	$1 - A_1A_3^{-1}$	$(1 - A_1A_3^{-1})S$

where $\ln B_t^* = \ln B_t - A_1A_t^{-1}\ln B_1$.

The first control card is a general title card. The second is the job control card. On this card, the first two fields are blank, indicating the program assumes the default values for DELT and SNW. The third field contains the number of plots (22) and the fourth field contains the number of variates (or equations). There are six independent variables; the maximum number of iterations is set at 30; the print option is set to zero to print the results at the last iteration only; the data is located on file 5; there is one format card; no column of 1's is added to the data; there are 11 additional comments cards; and the blank in the last field indicates that the initial estimates are calculated from the data.

Following the job control card are the 11 additional comments cards. Next is the data format card followed by the data itself.

The data are a portion of those collected for a growth and yield study of managed hardwood stands being made by the Northeastern Forest Experiment Station. The program was written in Fortran for an IBM 370/168.

A deck and listing of the computer program described in this publication is available on request with the understanding that the U.S. Department of Agriculture cannot assure its accuracy, completeness, reliability, or suitability for any other purpose than that reported. The recipient may not assert any proprietary rights thereto nor represent it to anyone as other than a Government-produced computer program. For cost information, please write the authors at the: Northeastern Forest Experiment Station, 370 Reed Road, Broomall, Pa. 19008.

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APPENDIX I

```
**************************************************************************
*** CONTROL CARDS
  1. TITLE CARD  FORMAT = (10A8)
  2. JOB CONTROL CARD
     COLS.  1-10 DELT = END ITERATIONS INDICATOR. WHEN THE
                        PROPORTIONATE CHANGE OF THE LOG LIKELIHOOD
                        FUNCTION IS LESS THAN DELT, ITERATIONS
                        TERMINATE. IF 0 IS READ, DELT IS SET TO 1.E-10
           11-20 SNW  = START NEWTON-RAPHSON PROCEDURE  WHEN THE
                        PROPORTIONATE CHANGE OF THE LOG LIKELIHOOD
                        FUNCTION IS LESS THAN *SNW*,  DEFAULT = .1  .
                        IF THE PROPORTIONATE CHANGE IS GREATER THAN
                        *SNW*, THE FISHER SCORING  PROCEDURE IS USED.
           21-25 N    = NO. OF EXPERIMENTAL UNITS (NO. OF PLOTS).
           26-30 P    = NO. OF REPEATED MEASUREMENTS OR DEPENDENT
                        VARIATES MEASURED ON EACH PLOT.
           31-35 Q    = NO. OF INDEPENDENT VARIABLES
           36-40 NITR = MAX. NO. ITERATIONS. (IF 0, NITR=20).
           41-45 IPR  = PRINT OPTION
                      = 0 PRINT RESULTS AT LAST ITERATION ONLY.
                      = 1 PRINT LOG LIKELIHOOD FUNCTION, DIFFERENCE,
                          AND VALUE OF QUADRATIC FORM EACH ITERATION.
                      = 2 PRINT EVERYTHING BUT NEG. OF INVERSE OF
                          MATRIX OF SECOND DERIVATIVES EACH ITERATION.
           46-50 F    = FILE ON WHICH DATA IS LOCATED (DEFAULT=5).
           51-55 NF   = NO. FORMAT CARDS  (MAX = 3).
           56-60 INT  = INTERCEPT INDICATOR; IF INT NOT EQUAL 0, THE
                        NO. OF INDEPENDENT VARS. IS INCREASED BY 1 AND
                        A COL. OF 1'S IS ADDED TO INDEP. VARS.
           61-65 NC   = NO.OF ADDITIONAL COMMENTS CARDS (CARD TYPE 3)
           66-70 INP  = INPUT INDICATOR.   IF  INP ...
                      = 0 INITIAL ESTIMATES OF BETA AND SIGMA ARE
                          CALCULATED FROM DATA
                      = 1 READ STARTING VALUES OF BETA AND SIGMA
                          (CARD TYPE 6 AND 7) PUNCHED ON PREVIOUS RUN.
  3. COMMENTS CARDS - OPTIONAL, READ ONLY IF *NC* ON CARD 2 IS .GT. 0.
           FORMAT = (10A8) - VARIABLE NAMES CAN BE READ HERE; I.E.
           Y1 = LN(VOL)       Y2 = LN(BA)      X1 = 1/AGE     ... ETC.
  4. FORMAT OF DATA DESCRIBING THE P*(Q+1) VARIABLES FOR A PLOT.
           (ALL VARIABLES ARE REAL)
  5. DATA  (1Y FOLLOWED BY Q X'S; THIS IS REPEATED P TIMES FOR
           THE  P  CORRELATED OBSERVATIONS ON 1 PLOT.  THERE ARE N
           OF THESE OBSERVATIONS)

     READ CARDS 6 AND 7 ONLY IF  INP  ON CARD 2 IS EQUAL TO 1.
  6. INITIAL ESTIMATES OF BETA,    FORMAT = (5D16.8)
  7. INITIAL ESTIMATES OF SIGMA (UPPER TRIANGULAR PORTION STORED
     COLUMNWISE - I.E. (1,1) (1,2) (2,2) (1,3) (2,3) (3,3) ...
     FORMAT = (5D16.8).

**************************************************************************
```

```
*************************************************************************
*** INSTRUCTIONS FOR CHANGING ARRAY DIMENSIONS. THIS IS REQUIRED ONLY
    IN MAIN ROUTINE.

    1. DETERMINE P AND Q.
    2. FROM THESE VALUES CALCULATE PP,PPP,QQ,MD1,MD2 USING ARRAY SIZE
       DEFINITIONS BELOW.
    3. CHANGE DIMENSION SIZE OF ARRAYS TO APPROPRIATE VALUES.
    4. CHANGE MXP TO P, MXQ TO Q, AND MXMD1 TO MD1.
*************************************************************************

*************************************************************************
*** ARRAY SIZE DEFINITIONS
    PP  =  P*(P+1)/2
    PPP=  PP*(PP+1)/2
    QQ  =  Q*(Q+1)/2
    MD1=  Q+PP                      MAX = MXMD1
    MD2=  MD1*(MD1+1)/2
*************************************************************************
*** ARRAY DIMENSIONS, MUST BE AT LEAST THE SIZE BELOW.
    REAL*8 DERIV2(MD2),SIGMA(PP),YPY(PP),SIGINV(PP),PLS(PP),
    DERIV1(MD1),PLBB(QQ),PLSS(PPP),DELTA(MD1),Y(P),X(P,Q),
    REAL*8 XPX(PP,Q,Q),XPY(P,Q,P),BETA(Q),PLBS(Q,PP),PLB(Q)
*************************************************************************

*************************************************************************
*** THERE ARE 4 PLACES WHERE FATAL MATRIX INVERSIONS MIGHT OCCUR.
    THESE ARE NUMBERED WITH  MIER. A MESSAGE IS PRINTED WITH THIS
    VALUE TO TELL THE USER THE MATRIX IN WHICH THE INVERSION FAILED.
    IF MIER = 1, THE INITIAL X'X COULD NOT BE INVERTED
              2, THE INITIAL ESTIMATE OF SIGMA COULD NOT BE INVERTED
              3, X'S X,THE MATRIX OF 2ND DERIVS. WITH RESPECT TO BETA
              4, X'S X, AT LAST ITERATION
*************************************************************************
```

APPENDIX II

```
SAMPLE PROBLEM,  GROWTH AND YIELD USING CLUTTER'S EQUATIONS
                              22    5    6   30    0    5    1    0    11
****************************************************************************
 SIMULTANEOUS EQUATIONS                          VARIABLE NAMES
Y1 = LN(VOL) AT TIME 1;     Y2 = LN(VOL) AT TIME 2;
Y3 = LN(VOL) AT TIME 3;     Y4 = (LN(BA2) - LN(BA1)*AGE1/AGE2);
Y5 = (LN(BA3) - LN(BA1)*AGE1/AGE3);    X1 = INTERCEPT;  X2 = SITE INDEX;
X3 = 1/(PLOT AGE); X4 = LN(BA(T)); X5 = (1-A1/A(T)); X6 = (1-A1/A(T))*SI
***     Y1,Y2,Y3 = F(X1,X2,X3,X4) ;   Y4,Y5 = F(X5,X6)        ***

              Y         X1   X2      X3         X4          X5         X6

****************************************************************************
(8X,F10.5,2F5.1,4F10.5)
 1 1 1 1    7.61415  1.0 66.0   0.02174    4.26816     0.0         0.0
 1 2 1 2    7.79565  1.0 66.0   0.01923    4.39371     0.0         0.0
 1 3 1 3    7.90747  1.0 66.0   0.01786    4.44811     0.0         0.0
 1 2 2 4    0.61803  0.0  0.0   0.0        0.0         0.11538    7.61539
 1 3 2 5    0.94212  0.0  0.0   0.0        0.0         0.17857   11.78572
 2 1 1 1    6.76869  1.0 60.0   0.02083    3.44829     0.0         0.0
 2 2 1 2    7.06133  1.0 60.0   0.01852    3.67579     0.0         0.0
 2 3 1 3    7.30458  1.0 60.0   0.01724    3.85767     0.0         0.0
 2 2 2 4    0.61065  0.0  0.0   0.0        0.0         0.11111    6.66667
 2 3 2 5    1.00391  0.0  0.0   0.0        0.0         0.17241   10.34483

                                  .

                                  .

                 .                .

21 1 1 1    7.16369  1.0 59.0   0.02564    3.96575     0.0         0.0
21 2 1 2    7.39018  1.0 59.0   0.02222    4.13613     0.0         0.0
21 3 1 3    7.58134  1.0 59.0   0.02041    4.27054     0.0         0.0
21 2 2 4    0.69914  0.0  0.0   0.0        0.0         0.13333    7.86667
21 3 2 5    1.11412  0.0  0.0   0.0        0.0         0.20408   12.04082
22 1 1 1    6.64747  1.0 57.0   0.02439    3.44319     0.0         0.0
22 2 1 2    6.92756  1.0 57.0   0.02128    3.65377     0.0         0.0
22 3 1 3    7.13688  1.0 57.0   0.01961    3.79762     0.0         0.0
22 2 2 4    0.65014  0.0  0.0   0.0        0.0         0.12766    7.27660
22 3 2 5    1.02957  0.0  0.0   0.0        0.0         0.15608   11.17647

SAMPLE PROBLEM,  GROWTH AND YIELD USING CLUTTER'S EQUATIONS

PROGRAM CONTROL INFORMATION

NO. PLOTS =                                22
NO. CORRELATED VARIATES =                   5
NO. INDEPENDENT VARIABLES =                 6
MAX. NO. ITERATIONS =                      30
PRINT OPTION =                              0
INPUT FILE =                                5
NO. FORMAT CARDS =                          1
INTERCEPT INDICATOR =                       0
DELTA (END ITERATIONS INDICATOR) =  0.100E-05
SNW (NEWTON RAPHSON CUTOFF POINT) = 0.100E+00

ADDITIONAL COMMENTS ABOUT VARIABLES OR OTHER PROBLEM   IDENTIFICATION
***************************************************************************
****************************************************************************
 SIMULTANEOUS EQUATIONS                       VARIABLE NAMES
Y1 = LN(VOL) AT TIME 1;     Y2 = LN(VOL) AT TIME 2;
Y3 = LN(VOL) AT TIME 3;     Y4 = (LN(BA2) - LN(BA1)*AGE1/AGE2);
Y5 = (LN(BA3) - LN(BA1)*AGE1/AGE3);    X1 = INTERCEPT;  X2 = SITE INDEX;
X3 = 1/(PLOT AGE); X4 = LN(BA(T)); X5 = (1-A1/A(T)); X6 = (1-A1/A(T))*SI
***     Y1,Y2,Y3 = F(X1,X2,X3,X4) ;   Y4,Y5 = F(X5,X6)      ***

              Y        X1   X2      X3         X4          X5          X6

****************************************************************************

DATA FORMAT
************
(8X,F10.5,2F5.1,4F10.5)

Y'Y ADJUSTED FOR MEAN
*********************
  1     0.1501314D+00
  2     0.1291906D+00   0.1120868D+00
  3     0.1058774D+00   0.9635714D-01   0.8354256D-01
  4    -0.5266364D-02  -0.4043155D-02  -0.3194546D-02   0.1184220D-02
  5    -0.1380618D-01  -0.1105377D-01  -0.8351621D-02   0.1832713D-02   0.3780428D-02

THE DETERMINANT OF CROSS PRODUCT MATRIX OF (Y - YBAR)  =      .8193308?D-14

INITIAL ESTIMATES OF REGRESSION COEFFICIENTS  (BETA)
**************************************************
  0.3409556?D+01  0.1195127D-01 -0.2951573D+02  0.9534858?D+00  0.4404521D+01  0.1515420?D-01

INITIAL ESTIMATE OF SIGMA,      NO. OF ADJUSTMENTS =  0
**************************************************
  1     0.2427787D-03
  2     0.1074727D-03   0.2736363D-03
  3     0.8128433D-04  -0.1158444D-04   0.5335672D-03
  4    -0.2187557D-03   0.2186?31D-03  -0.4157531D-03   0.1537300D-02
  5    -0.3976943D-03  -0.3639850D-03   0.1455963D-03   0.5265973D-03   0.2390865D-02

THE DETERMINANT OF SIGMA =     .1713627?D-16
*********************************************
```

```
SIGMA INVERSE
*************
 1   0.7720395D+04
 2  -0.3727898E+04   0.1999925D+05


 3  -0.2553490D+03  -0.2632553D+04   0.1530587D+04
 4   0.1412169E+04  -0.3713461D+04   0.1755342D+04   0.2277022E+04
 5   0.3620952E+03   0.2191034D+04  -0.1127965D+04  -0.1034345E+04   0.1151764E+04
THE VALUE OF THE QUADRATIC FORM =      110.00000
**************************************************

LOG OF THE LIKELIHOOD FUNCTION =     268.54785
**************************************************


ITERATION =  21       PROCEDURE USED AT THIS STEP = NEWTON-RAPHSON
**********************************************************************

LOG OF THE LIKELIHOOD FUNCTION =     294.76552        DIFFERENCE =     .2101330D-08      PROPORTIONATE DIFFERENCE =     .7370047E-11
********************************************************************************************************************************************

THE VALUE OF THE QUADRATIC FORM =      105.99597
**************************************************

REGRESSION COEFFICIENTS  (BETA)
******************************
 0.2975392D+01  0.1114751D-01 -0.1790866E+02  0.1002775D+01  0.4710562D+01  0.9703411D-02

ESTIMATE OF COVARIANCE MATRIX  (SIGMA)
**************************************
 1   0.1047714E-02
 2   0.1433310D-02   0.2003516D-02
 3   0.2552576E-02   0.3565800D-02   0.6577001D-02
 4  -0.5612056E-03  -0.7262475D-03  -0.1367971D-02   0.1171416E-02
 5   0.4215275D-03   0.1402584D-02   0.2142350D-02   0.5410953E-03   0.2595620E-02

DELTA (PARAMETER CHANGE),  NO. OF QUARTERING STEPS =  1      BETA(K), K=1,Q , SIGMA(I,J), (I=1,P;J=I,P)
************************************************************************************************************
 0.7488700-05   0.2054290-08 -0.2990120-05 -0.5517460-07  0.7070430-07 -0.6153380-10 -0.5975690-01  0.2600850-01  0.2554480-01
-0.1467170-01   0.1235890-01 -0.1638100-01 -0.9248070-02  0.3085020-02 -0.4310610-02 -0.6946990-02  0.4672110-02 -0.2345510-02
-0.2539520-02   0.1237690-02 -0.2493560-03

THE NORM OF THE DIFFERENCE =    0.7603143D-01
**************************************************

VECTOR OF FIRST DERIVATIVES;  BETA(K), K=1,Q  ,  SIGMA(I,J) (I=1,P;J=I,P)
***************************************************************************
-0.6676450-04 -0.4537160-07  0.1367460-06 -0.3103440-03 -0.7117140-05 -0.4873350-03  0.1394380-07  0.4147170-07  0.8077600-07
-0.1803120-07  0.3674010-07  0.3176700-07  0.1195780-06 -0.2659000-07  0.5434210-07  0.1159120-06 -0.5137200-07  0.1052220-05
 0.5606460-03 -0.2329980-07  0.2490970-07

THE DETERMINANT OF SIGMA =      .3931129D-17
**************************************************

INVERSE OF  (-1) * MATRIX OF EXPECTED VALUES OF SECOND DERIVATIVES WITH RESPECT TO BETA
*****************************************************************************************
 1   0.1769537D-02
 2  -0.1235532E-04   0.1525756D-06
 3  -0.1414655E-01   0.2786086D-04   0.2650173D+00
 4  -0.1505787E-03   0.3587289D-06   0.1157064D-02   0.2438530D-04
 5  -0.2533306E-02   0.4525079D-04   0.1774341D-01  -0.5159120E-04   0.2573517D+00
 6   0.4003267D-04  -0.6773418D-06   0.6638747D-04   0.2502355E-06  -0.3786591D-02   0.5622020D-04

INVERSE OF  (-1) * MATRIX OF EXPECTED VALUES OF SECOND DERIVATIVES WITH RESPECT TO SIGMA(I,J) , (I=1,P;J=I,P)
***************************************************************************************************************
 1   0.9946235E-07
 2   0.1358760E-06   0.1131131D-06


 3   0.2415301E-06   0.1346822D-06   0.6172917D-06
 4  -0.5309595E-07  -0.7091812D-07  -0.1190268D-06   0.6973107D-07
 5   0.8719502D-07   0.1263536D-06   0.2246811D-06   0.2086085E-08   0.1803263E-06
 6   0.1967616E-06   0.2613755D-06   0.4646265D-06  -0.9463059D-07   0.1827579E-06   0.3649596D-06
 7   0.3324329D-06   0.4647676D-06   0.8622205D-06  -0.1744876D-06   0.3219566E-06   0.6495057D-06   0.1176947D-05
 8  -0.7312571E-07  -0.5942686D-07  -0.1430853D-06   0.5986435D-07  -0.5264537E-09  -0.1322850D-06  -0.2422986D-06   0.1306503D-06

 9   0.1213882D-06   0.1753151D-06   0.3084983D-06   0.4423598E-08   0.2535355E-06   0.2554789D-06   0.4497687D-06   0.2978854D-06
     0.3627627E-06
10   0.5923310D-06   0.3274520D-06   0.1527208D-05  -0.3175405D-06   0.5667528E-06   0.1155902D-05   0.2132024D-05  -0.4434465D-06
     0.7917257D-06   0.3932448D-05
11  -0.1332243E-06  -0.1752250D-06  -0.3264557D-06   0.1708110E-06   0.4782702E-09  -0.2354230D-06  -0.4388382D-06   0.2350236D-06
     0.7376622E-08  -0.8179225D-06   0.4352612D-06
12   0.2138646E-06   0.3121146D-06   0.5586596D-06   0.5974251E-08   0.4499081E-06   0.4546666D-06   0.8151692D-06   0.4883003D-09
     0.6412768D-06   0.1460301D-05   0.4896155D-08   0.1166753D-06
13   0.2863206E-07   0.3705225D-07   0.6979221D-07  -0.5576415E-07  -0.2766601E-07   0.4794366D-07   0.9931687D-07  -0.7733979D-07
    -0.3572446D-07   0.1701223D-06  -0.1456785D-06  -0.6729110E-07   0.1247468E-06
14  -0.4702004E-07  -0.6207950D-07  -0.1196092D-06   0.3576986E-07  -0.5375399E-07  -0.9260208D-07  -0.1678384D-06   0.5682000D-07
    -0.6435884E-07  -0.3137331D-06   0.6460170D-07  -0.1262114E-06   0.5762250E-07   0.1728243D-06
15   0.7721706D-07   0.1175139D-06   0.2346294D-06   0.4533504D-07   0.2510013E-06   0.1788401D-06   0.3114182D-06   0.5899377D-07
     0.3819896E-06   0.5422791D-06   0.1221403D-06   0.6651670E-06   0.2661673E-07   0.1473658D-06   0.8159030D-06

COVARIANCE MATRIX IN CORRELATION FORM
**************************************
 1   0.1000000E+01
 2   0.9925789E+00   0.1000000D+01
 3   0.9756610E+00   0.9322762D+00   0.1000000D+01
 4  -0.5082781D+00  -0.4740451D+00  -0.4928418D+00   0.1000000E+01
 5   0.5719507E+00   0.5724818D+00   0.5502194D+00   0.2889420E+00   0.1000000E+01
```